

ON THE PROBLEM BY ERDÖS-DE BRUIJN-KINGMAN ON REGULARITY OF RECIPROALS FOR EXPONENTIAL SERIES

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ABSTRACT. Motivated by applications to renewal theory, Erdős, de Bruijn and Kingman posed in 50th-70th a problem on regularity of reciprocals of probability generating functions. We solve the problem in the strong negative and give a number of other related results.

1. INTRODUCTION

1.1. Motivation. The paper is devoted to solutions of certain problems related to renewal sequences and their generating functions. Recall that if $(a_k)_{k=1}^\infty$ is such that $a_k \geq 0, k \geq 1$, and $\sum_{k \geq 1} a_k = 1$ then the sequence $(b_k)_{k=0}^\infty$ given by the recurrence relation

$$(1.1) \quad b_n = \sum_{k=1}^n a_k b_{n-k}, \quad b_0 = 1, \quad n \in \mathbb{N},$$

is called the renewal sequence associated to $(a_k)_{k=1}^\infty$. Renewal sequences is a classical object of studies in probability theory, in particular, in the theory of Markov processes. To mention one of the probabilistic meanings of (1.1), note that given a discrete Markov chain, (1.1) expresses the diagonal transition probabilities $(b_k)_{k=0}^\infty$ in terms of the recurrence time probabilities $(a_k)_{k=1}^\infty$.

Moreover, the renewal sequences are of substantial interest in ergodic theory. For the applications in ergodic theory one may consult e.g. the papers [1], [2] and [16], the book [3] and the references therein.

It is often convenient to study $(a_k)_{k=1}^\infty$ and $(b_k)_{k=0}^\infty$ in terms of their generating functions F and G given by

$$F(z) = \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad G(z) = \sum_{k=0}^{\infty} b_k z^k.$$

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The functions are defined in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and connected by the relation

$$G(z) = \frac{1}{1 - F(z)}.$$

Being unable to give any account of the wide topic of renewal sequences we refer to the classical sources such as for instance [21], [30], and [13] (although the term “renewal sequence” for $(b_n)_{n=0}^\infty$ given by (1.1) is used only in [21]).

1.2. History. One of the first and foundational results in theory of renewal sequences is the famous Erdős-Feller-Pollard theorem. To recall it we need to introduce certain notation. Let \mathcal{A}^+ consist of the power series of the form

$$(1.2) \quad F(z) = \sum_{k=1}^{\infty} a_k z^k, \quad a_k \geq 0, \quad \sum_{k=1}^{\infty} a_k = 1$$

in $\overline{\mathbb{D}}$. It is a complete metric space with metric induced by ℓ_1 -norm on an appropriate sequence space. We say that $F \in \mathcal{A}^+$ is aperiodic if $F(z) = 1, z \in \overline{\mathbb{D}}$, implies that $z = 1$.

Using Wiener’s theorem, it was proved in [11] that if $F \in \mathcal{A}^+$ is aperiodic and additionally

$$(1.3) \quad \mu := \sum_{k=1}^{\infty} k a_k < \infty,$$

then

$$(1.4) \quad \lim_{k \rightarrow \infty} b_k = 1/\mu.$$

This is essentially the famous Erdős-Feller-Pollard theorem, one of the first and basic limit theorems in the renewal theory.

The key point in [11] for showing the property (1.4) was the fact that the function $(1 - z)(1 - F)^{-1}$ has absolutely convergent Taylor series:

$$(1.5) \quad \sum_{k=1}^{\infty} |b_k - b_{k+1}| < \infty.$$

The theorem generated an area of research, and a huge number of its generalizations and improvements in various directions has appeared in subsequent years. Analytic approaches to the study of $1/(1 - F)$ and of asymptotics of $(b_k)_{k=1}^\infty$ are discussed e.g. in [22, Chapter V.22] and [27, Chapter 24]. These books contain a number of related references. We mention here only the classical papers [31] and [12].

However, certain natural questions have been left open to the best of our knowledge. In particular, P. Erdős and N. de Bruijn suggested in [7, p. 164] that (1.5) is probably true for any aperiodic $F \in \mathcal{A}^+$ and the assumption (1.3) is redundant. As they wrote in [7], “it seems possible that the condition (1.3) is superfluous”. Moreover, the question whether (1.5) holds for any aperiodic F satisfying (1.2) was formulated as an open

problem by J. Kingman in [21, p. 20-21, (iv)]. A recent discussion of the problem in the context of ergodic theory can be found in [2]. The analysis of $(1-z)(1-F)^{-1}$ presents certain difficulties in view of nonlinear character of the transformation $F \mapsto (1-F)^{-1}$. While $(b_k)_{k=0}^\infty$ is given explicitly in terms of $(a_k)_{k=1}^\infty$, it is very difficult to study it by means of the recurrence relation (1.1) (see e.g. [6] and [7] for such a direct approach). So most of research on analytic properties of renewal sequences concentrated on the generating functions methodology.

One must note that relevant studies has been made by J. Littlewood in [25], a paper apparently overlooked by mathematical community. Being motivated by the enigmatic message from Besicovitch (see [26, p. 145]) and a question by W. Smith, Littlewood proved in [25] that for any function f given by

$$(1.6) \quad f(\theta) = \sum_{k=1}^{\infty} a_k e^{i\lambda_k \theta}, \quad \text{where } \lambda \in \mathbb{R}, \lambda_k \geq 1, \lim_{k \rightarrow \infty} \lambda_k = \infty,$$

and $(a_k)_{k=1}^\infty$ as in (1.2), one has

$$(1.7) \quad \overline{\lim}_{\theta_0 \rightarrow 0} \int_{\theta_0}^{2\theta_0} \frac{d\theta}{|1 - f(e^{i\theta})|} < \infty.$$

(Sometimes f satisfying (1.6) are called quasi-exponential series.) In particular, there is $\delta > 0$ (depending on f) such that

$$(1.8) \quad \int_0^\delta \frac{\theta^\alpha d\theta}{|1 - f(e^{i\theta})|} < \infty$$

for any $\alpha > 0$. The results of that type lead to a number of useful consequences in the study of regularity for generating functions of renewal sequences as we show in Section 5.

It is natural to ask whether Littlewood's results can essentially be improved. For example, the boundedness of $|1 - f(e^{i\theta})|/\theta$ in the neighborhood of zero would imply (1.7). Littlewood's student H. T. Croft claimed in [8] that the latter property does not hold, in general. More precisely, if f is defined by (1.6) then for any function χ such that $\chi(\theta) \uparrow \infty$ as $\theta \rightarrow 0$ there exist sequences $(a_k)_{k=1}^\infty$ and $(\lambda_k)_{k=1}^\infty$ as above, and $(\theta_k)_{k=1}^\infty$ satisfying $\theta_k \rightarrow 0$ as $k \rightarrow \infty$ such that

$$(1.9) \quad |1 - f(\theta_k)| \leq \chi(\theta_k) \theta_k^2, \quad k \in \mathbb{N}.$$

(In fact, only the case $\chi(\theta) = \theta^{-\epsilon}$ was discussed in [8].) This, indirectly, would solve the Erdős-de Bruijn-Kingman problem once one would arrange the integer frequencies λ_k above, although Croft presumably was not aware of the problem. However, [8] contains only a hint rather than a complete argument, and it produces merely real frequencies λ_k rather than integer ones as in (1.2).

It is also instructive to remark that in [17] J. Hawkes constructed a lacunary series of the form (1.6) with $\theta_k = 2^{-2k^2}$ and $\lambda_k = \frac{2\pi - \theta_k^{2/3}}{\theta_k}$, $k \in \mathbb{N}$, such

that

$$\lim_{k \rightarrow \infty} \frac{|1 - f(\theta_k)|}{\theta_k} = 0.$$

This way, Hawkes solved another Kingman's problem formulated in [21, p. 76], which is similar (but not equivalent) to the problem mentioned above.

1.3. Results. In this paper, we answer the question by Erdős, de Bruijn and Kingman in the *strong negative*. Namely, we prove in Theorem 4.3 that for any positive sequence $(\epsilon_k)_{k=1}^\infty$ tending to zero (subject to a technical assumption) there exists an aperiodic $F \in \mathcal{A}^+$ with

$$(1.10) \quad \sum_{k=1}^{\infty} k \epsilon_k a_k < \infty,$$

such that $(1 - z)(1 - F)^{-1}$ is not even bounded in \mathbb{D} , and thus (1.5) is not true. Moreover, the set of such F is dense in \mathcal{A}^+ (when \mathcal{A}^+ is considered as a metric space with a natural metric). Thus, the assumption (1.3) in the Erdős-Feller-Pollard theorem is best possible as far as the “smoothness” of $(b_k)_{k=0}^\infty$ is concerned. Several results of a similar nature have been obtained as well. At the same time, we show in Appendix B that Croft's idea can successfully be realized, and moreover it can also be realized for the integer frequencies.

Our technique is based on constructing special sequences of polynomials approximating well enough a given polynomial in an appropriate norm and, as in (1.9), the constant function 1 at a sequence of points from the unit circle converging to 1. By means of either Baire category arguments or inductive reasoning, this then turns into the same estimates for exponential series

$$f(\theta) = F(e^{i\theta}), \quad |\theta| \leq \pi, \quad F \in \mathcal{A}^+.$$

It is crucial that the bounds of the type (1.9) can also be spread out to an appropriate sequence of intervals approaching 1, and thus hold on a set of sufficiently large measure. These extended bounds generalize the upper estimates from [8] and [17], and they allow us to get rid of a certain amount of regularity of $(1 - F)^{-1}$, e.g. with respect to the L^p -scale.

By pursuing our studies a bit further, it is natural to ask what kind regularity is possessed by $(1 - F)^{-1}$ *without any a priori assumptions* on the sequence of Taylor coefficients $(a_k)_{k=1}^\infty$ of F . Despite enormous number of papers on renewal sequences, the question seems to have not been adequately addressed so far (apart probably to some extent [1], [2] and [25]). In the present paper, we make several steps in this direction. First, we extend Littlewood's results (1.7) and (1.8) by relating the integrability of $(1 - F)^{-1}$ on an interval $(\theta_0, 2\theta_0) \subset (0, 2)$ to the summability properties of the Taylor coefficients of F . This allows us to obtain sharp and explicit conditions for the integrability of $(1 - F)^{-1}$ on $[-\pi, \pi]$ if F is aperiodic. Furthermore, we pursue a similar study for the “smoothed” function $(1 - z)(I - F)^{-1}$

appearing in the Erdős-de Bruijn-Kingman problem. We show that for F as in (1.2), satisfying

$$(1.11) \quad \sum_{k=1}^{\infty} k^{\nu} a_k < \infty$$

for some $\nu \in (0, 1)$, one has

$$(1 - z)(1 - F)^{-1} \in L_{1+1/(1-\nu)}[-\pi, \pi].$$

On the other hand, for each $p \in (2 + (1 - \nu)^{-1}, \infty)$ we construct a function F_p of the form (1.2) satisfying (1.11) but at the same time violating

$$(1 - z)(1 - F_p)^{-1} \notin L^p[-\pi, \pi].$$

Remark that while (1.5) is not, in general, true for $F \in \mathcal{A}^+$ (as we show in this paper) we prove that nevertheless a weaker property holds:

$$\sum_{k=0}^{\infty} (b_k - b_{k+1})^2 < \infty.$$

This simple result has probably been overlooked in the literature. Moreover, we show that, in general, $(1 - z)(1 - F)^{-1} \notin L^p[-\pi, \pi]$ if $p \in (3, \infty)$. The problem what happens if $p \in (2, 3]$ remains, unfortunately, open.

2. PRELIMINARIES AND NOTATIONS

For $w = (w_k)_{k=1}^{\infty} \subset (1, \infty)$ we denote by $\mathcal{A}(w)$ a Banach space

$$\left\{ f(\theta) = \sum_{k=1}^{\infty} a_k e^{ik\theta} : \sum_{k=1}^{\infty} w_k |a_k| < \infty, \quad |\theta| \leq \pi \right\},$$

with the norm

$$\|f\|_{\mathcal{A}(w)} = \sum_{k=1}^{\infty} w_k |a_k|, \quad f \in \mathcal{A}(w).$$

Its subset $\mathcal{A}^+(w)$ given by

$$\mathcal{A}^+(w) = \left\{ \sum_{k=1}^{\infty} a_k e^{ik\theta} \in \mathcal{A}(w) : a_k \geq 0, \quad \sum_{k=1}^{\infty} a_k = 1 \right\}$$

is a complete metric space with the metric $\rho(\cdot, \cdot)_{\mathcal{A}(w)}$ inherited from $\mathcal{A}(w)$.

Note that

$$(2.1) \quad |f(\theta) - g(\theta)| \leq \|f - g\|_{\mathcal{A}(w)}, \quad f, g \in \mathcal{A}^+(w), \quad |\theta| \leq \pi.$$

We will often be using a more intuitive notation $\|f_1 - f_2\|_{\mathcal{A}(w)}$ instead of $\rho(f_1, f_2)_{\mathcal{A}(w)}$ whenever it is defined correctly. If $w_k = k^{\nu}$, $\nu \in [0, 1)$, for $k \in \mathbb{N}$, then we will write $\mathcal{A}(\nu)$ instead of $\mathcal{A}(w)$ slightly abusing our notation. We will also write \mathcal{A}^+ (respectively \mathcal{A}) instead of $\mathcal{A}^+(\{1\})$ (respectively $\mathcal{A}(\{1\})$).

In the sequel, we identify absolutely convergent power series F on $\overline{\mathbb{D}}$ with their boundary values on \mathbb{T} , and the boundary values with the corresponding 2π -periodic functions f , so that

$$(2.2) \quad f(\theta) := F(e^{i\theta}) = \sum_{k=1}^{\infty} a_k e^{ik\theta}, \quad |\theta| \leq \pi.$$

Let us recall that $f \in \mathcal{A}^+$ is aperiodic if and only if the greatest common factor of $\{n \in \mathbb{N} : a_n > 0\}$ is 1, see e.g. [30, p. 85] or [13, Vol II, p. 500-501]. In particular, any function $f \in \mathcal{A}^+$ of the form (2.2) with $a_1 \neq 0$, is aperiodic.

Observe that the set of aperiodic polynomials from $\mathcal{A}^+(w)$ is dense in $\mathcal{A}^+(w)$. Indeed, let $f \in \mathcal{A}^+(w)$ be given by

$$(2.3) \quad f(\theta) = \sum_{k=m}^{\infty} a_k e^{ik\theta}, \quad a_m \neq 0.$$

Let us define for $n \geq m+1$ the family of aperiodic polynomials

$$P_n(\theta) = \left(\frac{a_m}{n} e^{i\theta} + a_m \left(1 - \frac{1}{n}\right) e^{im\theta} + \sum_{k=m+1}^n a_k e^{ik\theta} \right) / d_n \in \mathcal{A}^+,$$

where

$$d_n = \sum_{k=m}^n a_k \rightarrow 1, \quad n \rightarrow \infty.$$

Then

$$\begin{aligned} \|f - P_n\|_{\mathcal{A}(w)} &\leq (1/d_n - 1) \|f\|_{\mathcal{A}(w)} + d_n^{-1} \|f - d_n P_n\|_{\mathcal{A}(w)} \\ &\leq (1/d_n - 1) \|f\|_{\mathcal{A}(w)} + \frac{(w_1 + w_m)a_m}{nd_n} \\ &\quad + \frac{1}{d_n} \sum_{k=n+1}^{\infty} w_k a_k \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

The next simple proposition will be useful for the sequel. It is probably known, but we were not able to find an appropriate reference.

Proposition 2.1. *The set of aperiodic functions in $\mathcal{A}^+(w)$ is open in $\mathcal{A}^+(w)$.*

Proof. Let $(f_n)_{n=1}^{\infty} \subset \mathcal{A}^+(w)$ be a sequence of non-aperiodic functions such that

$$(2.4) \quad \lim_{n \rightarrow \infty} \|f_0 - f_n\|_{\mathcal{A}(w)} = 0.$$

Note that for every $n \in \mathbb{N}$ there exists $\theta_n \in [\pi/2, \pi]$ such that $f_n(\theta_n) = 1$. If θ_0 is any limit point of $(\theta_n)_{n=1}^{\infty}$ then $\theta_0 \in [\pi/2, \pi]$, and from (2.4), (2.1) and the continuity of f_0 it follows that $f_0(\theta_0) = 1$. Therefore, f_0 is not aperiodic, and the set of non-aperiodic functions is closed in $\mathcal{A}^+(w)$. \square

Remark 2.2. By Proposition 2.1 the set of aperiodic functions in $\mathcal{A}^+(w)$ is open in $\mathcal{A}^+(w)$. Since that set is also dense in $\mathcal{A}^+(w)$ as we showed above, the set of aperiodic functions in $\mathcal{A}^+(w)$ is residual.

Finally, we will fix some standard notation for the rest of the paper. For any measurable set $E \subset \mathbb{R}$ (or $E \subset \mathbb{T}$) we let $\text{meas}(E)$ stand for its Lebesgue measure. A usual max norm in the space of 2π -periodic continuous functions on $[-\pi, \pi]$ will be denoted by $\|\cdot\|_\infty$. For an exponential polynomial $P \in \mathcal{A}^+$ its degree will be denoted by $\deg P$. Sometimes, to simplify the exposition, the constants will change from line to line, although in several places we will give the precise values of constants to underline their (in-)dependence on parameters.

3. AUXILIARY ESTIMATES OF THE EXPONENTIAL POLYNOMIALS

In this section, we first obtain the lower estimates for the size of approximations of the constant function 1 by exponential polynomials. Then in the next section these estimates will be extended to exponential series by either Baire category arguments or inductive constructions.

We start with following technical lemma.

Lemma 3.1. *For $\lambda \in (0, 1]$ and $\gamma \in (0, 1]$ define*

$$d_{\lambda, \gamma} = \frac{\sin(\lambda/2) \cos((\gamma + \lambda)/2)}{\sin(\gamma/2)}.$$

Then

$$(3.1) \quad \frac{\lambda}{4\gamma} \leq d_{\lambda, \gamma} \leq \frac{\lambda}{\gamma} \quad \text{and} \quad |(1 - e^{i\lambda}) + d_{\lambda, \gamma}(1 - e^{-i\gamma})| \leq \frac{\lambda(\gamma + \lambda)}{2}.$$

Since the proof of Lemma 3.1 is based on simple computations with trigonometrical functions, it will be postponed to Appendix A.

The next corollary gives a recipe for constructing exponential polynomials (having, in general, non-integer frequencies) with control of their size at a fixed point and of their variation on the unit circle.

Corollary 3.2. *Let $P(\theta) = \sum_{k=1}^n a_k e^{ik\theta} \in \mathcal{A}^+$. For all $\theta_0 \in (0, 1/n]$ and $\gamma \in (0, 1]$ there exists $d \in \left[\frac{\theta_0}{4\gamma}, \frac{n\theta_0}{\gamma}\right]$ such that if*

$$(3.2) \quad P_{d, \gamma}(\theta) := \sum_{k=1}^n \frac{a_k}{1+d} e^{ik\theta} + \frac{d}{1+d} e^{i(2\pi - \gamma)\frac{\theta}{\theta_0}},$$

then

$$(3.3) \quad |1 - P_{d, \gamma}(\theta_0)| \leq 2n\theta_0\gamma \quad \text{and} \quad \|P'_{d, \gamma}\|_\infty \leq n \left(1 + \frac{2\pi}{\gamma}\right).$$

Proof. Let $\theta_0 \in (0, 1/n]$, and $\gamma \in (0, 1]$ be fixed. Set

$$d := \sum_{k=1}^n a_k d_{k\theta_0, \gamma},$$

where $d_{k\theta_0, \gamma}$, $1 \leq k \leq n$, are given by Lemma 3.1. Then, by Lemma 3.1,

$$(3.4) \quad \frac{\theta_0}{4\gamma} \leq \frac{1}{4\gamma} \sum_{k=1}^n k a_k \theta_0 \leq d \leq \frac{1}{\gamma} \sum_{k=1}^n k a_k \theta_0 \leq \frac{n\theta_0}{\gamma}.$$

Note that

$$(1+d)(1-P_{d,\gamma}(\theta_0)) = \sum_{k=1}^n a_k [(1-e^{ik\theta_0}) + (1-e^{-i\gamma})].$$

So using (3.1) and (3.4), we obtain that

$$\begin{aligned} (1+d)|1-P_{d,\gamma}(\theta_0)| &\leq \sum_{k=1}^n a_k |(1-e^{ik\theta_0}) + d_{k\theta_0, \gamma}(1-e^{-i\gamma})| \\ &\leq \frac{1}{2} \sum_{k=1}^n k a_k \theta_0 (\gamma + k\theta_0) \\ &\leq \frac{n\theta_0\gamma}{2} \sum_{k=1}^n a_k \left(1 + \frac{k\theta_0}{\gamma}\right) \\ &\leq 2n\theta_0\gamma(1+d), \end{aligned}$$

hence the first estimate in (3.3) holds.

Finally, by (3.4),

$$\|P'_{d,\gamma}\|_\infty \leq \sum_{k=1}^n k a_k + d \frac{2\pi - \gamma}{\theta_0} \leq n + \frac{2\pi n\theta_0}{\theta_0\gamma} = n \left(1 + \frac{2\pi}{\gamma}\right),$$

i.e. the second estimate in (3.3) is true. \square

Now we are able to show that for any polynomial from \mathcal{A}^+ there is another polynomial close to it in an appropriate weighted norm and close to the function 1 on a sequence of points of \mathbb{T} going to 1.

Theorem 3.3. *Let $(\epsilon_k)_{k=1}^\infty$ be a positive sequence such that*

$$\lim_{k \rightarrow \infty} \epsilon_k = 0 \quad \text{and} \quad k\epsilon_k \geq 1, \quad k \in \mathbb{N},$$

and let $\tilde{w} = (k\epsilon_k)_{k=1}^\infty$. Then for every polynomial $P \in \mathcal{A}^+$ there exist a sequence $(\theta_m)_{m=1}^\infty$ decreasing to zero and a sequence of polynomials $(Q_m)_{m=1}^\infty \subset \mathcal{A}^+$ satisfying

$$\lim_{m \rightarrow \infty} \frac{|1 - Q_m(\theta_m)|}{\theta_m} = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \|P - Q_m\|_{\mathcal{A}(\tilde{w})} = 0.$$

Proof. Let a polynomial P be fixed, and let $\deg P = n$. Define $e_n := \sup\{\epsilon_k : 1 \leq k \leq n\}$, and choose a subsequence $(\epsilon_{s_m})_{m \geq 1}$ such that

$$\lim_{m \rightarrow \infty} \epsilon_{s_m} = 0.$$

Fix an integer $m > 2\pi n$ such that

$$(3.5) \quad \gamma_m := \left(\epsilon_{s_m} + \frac{ne_n}{s_m} \right)^{1/2} \leq 1$$

and put

$$\theta_m := \frac{2\pi - \gamma_m}{s_m}.$$

Using Corollary 3.2 with $\theta_0 = \theta_m$ and $\gamma = \gamma_m$ we conclude that there exist d_m , $0 < d_m \leq n\theta_m/\gamma_m$, and a polynomial $Q_m := P_{d_m, \gamma_m} \in \mathcal{A}^+$, $\deg Q_m = s_m > n$, defined by

$$Q_m(\theta) = \sum_{k=1}^n b_k e^{ik\theta} + b_{s_m} e^{is_m\theta},$$

such that

$$\frac{|1 - Q_m(\theta_m)|}{\theta_m} \leq 2n\gamma_m \quad \text{and} \quad \|P - Q_m\|_{\mathcal{A}} = \frac{2d_m}{1 + d_m} \leq \frac{2n\theta_m}{\gamma_m}.$$

Hence (3.5) and the latter inequality imply that

$$\|P - Q_m\|_{\mathcal{A}(\tilde{w})} \leq \max(ne_n, s_m\epsilon_{s_m}) \|P - Q_m\|_{\mathcal{A}} \leq s_m\gamma_m^2 \frac{2n\theta_m}{\gamma_m} \leq 4\pi n\gamma_m.$$

Since $\gamma_m \rightarrow 0$ as $m \rightarrow \infty$, the statement follows. \square

Remark 3.4. Here and the sequel, the assumption $k\epsilon_k \geq 1, k \in \mathbb{N}$, is of purely technical nature and has been made to simplify our exposition.

Recall from Section 2 that the set of polynomials in $\mathcal{A}^+(w)$ is dense in $\mathcal{A}^+(w)$ for any weight w , thus Theorem 3.3 implies the following statement.

Corollary 3.5. *Let $(\epsilon_k)_{k=1}^\infty \subset (0, \infty)$ satisfy $\lim_{k \rightarrow \infty} \epsilon_k = 0$ and $k\epsilon_k \geq 1$, $k \in \mathbb{N}$, and let $\tilde{w} = (k\epsilon_k)_{k=1}^\infty$. Then for every $f \in \mathcal{A}(\tilde{w})$ there exists a sequence of polynomials $(Q_m)_{m=1}^\infty \in \mathcal{A}^+$ such that*

$$\lim_{m \rightarrow \infty} \inf_{\theta \in (0, 1/m]} \frac{|1 - Q_m(\theta)|}{\theta} = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \|f - Q_m\|_{\mathcal{A}(\tilde{w})} = 0.$$

The next result is our basic statement allowing one to spread out the upper estimates for $|1 - Q_m|$ proved in Theorem 3.3 from the sequence $(\theta_m)_{m=1}^\infty$ to a larger set containing it. The result will help us to provide counterexamples on L^p -integrability of $(1 - z)(1 - F)^{-1}$.

Theorem 3.6. *Let $\psi : (0, 1] \mapsto (0, \infty)$ and $\chi : (0, 1] \mapsto (0, \infty)$ be continuous functions satisfying*

$$(3.6) \quad \lim_{\theta \rightarrow 0} \psi(\theta) = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \frac{\chi(\theta)}{\psi(\theta)} = 0.$$

Then for every polynomial $P \in \mathcal{A}^+$ there exist a sequence $(\theta_m)_{m=1}^\infty$ decreasing to zero and a sequence of polynomials $(Q_m)_{m=1}^\infty \subset \mathcal{A}^+$ such that for all $m \in \mathbb{N}$:

- (i) $m\theta_m \leq 2\pi$,
- (ii) $|1 - Q_m(\theta_m)| \leq 2\psi(\theta_m)\theta_m$,
- (iii) $\|P - Q_m\|_{\mathcal{A}} \leq \frac{2\theta_m}{\chi(\theta_m)}$.

Moreover, for each $m \in \mathbb{N}$ and for each θ such that $|\theta - \theta_m| \leq \psi(\theta_m)\chi(\theta_m)\theta_m$ one has

$$|1 - Q_m(\theta)| \leq 10\psi(\theta_m)\theta_m.$$

Proof. Let a polynomial P be fixed, and let $\deg P = n$. Making use of (3.6), choose $\Theta_0 \in (0, 1]$ in such a way that

$$(3.7) \quad \chi(\theta)\psi(\theta) + n \left(\frac{\chi^{1/2}(\theta)}{\psi^{1/2}(\theta)} + \theta \right) \leq 1, \quad \theta \in (0, \Theta_0].$$

Define

$$\tau(\theta) := \frac{2\pi - \gamma(\theta)}{\theta}, \quad \gamma(\theta) := \psi^{1/2}(\theta)\chi^{1/2}(\theta), \quad \theta \in (0, \Theta_0],$$

and note that, in particular, $n\theta \leq 1$, and $\gamma(\theta) \leq 1$ for all $\theta \in (0, \Theta_0]$.

Moreover, if $\theta \in (0, \Theta_0]$, then from (3.7) it follows that

$$(3.8) \quad n\gamma(\theta) \leq \frac{\psi^{1/2}(\theta)\gamma(\theta)}{\chi^{1/2}(\theta)} = \psi(\theta), \quad \frac{n}{\gamma(\theta)} \leq \frac{\psi^{1/2}(\theta)}{\gamma(\theta)\chi^{1/2}(\theta)} = \frac{1}{\chi(\theta)}.$$

Since γ is continuous on $(0, \Theta_0]$ and $\lim_{\theta \rightarrow 0} \tau(\theta) = +\infty$, there exists a sequence $(\theta_m)_{m \geq m_0} \subset [0, \Theta_0]$ satisfying

$$\tau(\theta_m) = m, \quad m \geq m_0.$$

Moreover, as $\lim_{\theta \rightarrow 0} \gamma(\theta) = 0$, we may also assume that $m\theta_m \leq 2\pi$.

Next, we fix $m \geq \max(m_0, n)$, set $\theta_0 = \theta_m$ and $\gamma = \gamma(\theta_m)$, and apply Corollary 3.2 to the polynomial P . Taking into account (3.8), we infer that there exist $d_m > 0$ and a polynomial $Q_m = P_{d_m, \gamma(\theta_m)} \in \mathcal{A}^+$ such that

$$|1 - Q_m(\theta_m)| \leq 2n\gamma(\theta_m)\theta_m \leq 2\psi(\theta_m)\theta_m,$$

and

$$\|P - Q_m\|_{\mathcal{A}} \leq \frac{2n\theta_m}{\gamma(\theta_m)} \leq \frac{2\theta_m}{\chi(\theta_m)}.$$

Moreover, by (3.3) and (3.8),

$$(3.9) \quad \|Q'_m\|_{\infty} \leq n \left(1 + \frac{2\pi}{\gamma(\theta_m)} \right) \leq \frac{\gamma(\theta_m)}{\chi(\theta_m)} \left(1 + \frac{2\pi}{\gamma(\theta_m)} \right) \leq \frac{(1 + 2\pi)}{\chi(\theta_m)}.$$

Using

$$(3.10) \quad |1 - Q_m(\theta)| \leq |1 - Q_m(\theta_m)| + \int_{\theta_m}^{\theta} |Q'_m(s)| ds,$$

and (3.9), we conclude that if $|\theta - \theta_m| \leq \psi(\theta_m)\chi(\theta_m)\theta_m$, then

$$|1 - Q_m(\theta)| \leq 2\psi(\theta_m)\theta_m + \frac{1 + 2\pi}{\chi(\theta_m)}|\theta - \theta_m| \leq 10\psi(\theta_m)\theta_m.$$

□

Remark 3.7. Let $P \in \mathcal{A}^+$ and let $Q_m \in \mathcal{A}^+$ be polynomials given by Theorem 3.6. If $(w_k)_{k=1}^\infty \subset (1, \infty)$ is a nondecreasing sequence then the estimate from Theorem 3.6, (iii) yields

$$\|P - Q_m\|_{\mathcal{A}(w)} \leq w_m \|P - Q_m\|_{\mathcal{A}} \leq \frac{2\theta_m w_m}{\chi(\theta_m)}.$$

In particular, if $w_k = k^\nu$, $k \in \mathbb{N}$, for some $\nu \in (0, 1)$, then in view of $\theta_m m \leq 2\pi$, $m \in \mathbb{N}$, one has

$$\|P - Q_m\|_{\mathcal{A}(\nu)} \leq \frac{2\theta_m m^\nu}{\chi(\theta_m)} \leq 2(2\pi)^\nu \frac{\theta_m^{1-\nu}}{\chi(\theta_m)}.$$

It will be convenient to separate the next easy corollary of Theorem 3.6

Corollary 3.8. *Let $\nu \in [0, 1]$ and let $\varphi : (0, 1] \mapsto (0, \infty)$ be a continuous function such that*

$$(3.11) \quad \lim_{\theta \rightarrow 0} \theta^{1-\nu} \varphi(\theta) = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \varphi(\theta) = \infty.$$

Then for every polynomial $P \in \mathcal{A}^+$ there exist a sequence $(\theta_m)_{m=1}^\infty$ decreasing to zero and a sequence of polynomials $(Q_m)_{m=1}^\infty \subset \mathcal{A}^+$ such that for all $m \in \mathbb{N}$:

- (i) $\theta_m m \leq 2\pi$,
- (ii) $|1 - Q_m(\theta_m)| \leq 2\varphi(\theta_m)\theta_m^{2-\nu}$,
- (iii) $\|P - Q_m\|_{\mathcal{A}(\nu)} \leq \frac{2(2\pi)^\nu}{\varphi^{1/2}(\theta_m)}.$

Moreover, for each $m \in \mathbb{N}$ and each θ such that $|\theta - \theta_m| \leq \varphi^{3/2}(\theta_m)\theta_m^{3-2\nu}$ one has

$$|1 - Q_m(\theta)| \leq 10\varphi(\theta_m)\theta_m^{2-\nu}.$$

Proof. Define

$$\psi(\theta) := \varphi(\theta)\theta^{1-\nu} \quad \text{and} \quad \chi(\theta) := \varphi^{1/2}(\theta)\theta^{1-\nu}, \quad \theta \in (0, 1].$$

Since χ and ψ satisfy (3.6), the corollary follows from Theorem 3.6 and Remark 3.7. \square

By the density arguments, the next result follows directly from Corollary 3.8.

Corollary 3.9. *Let $\varphi : (0, 1] \mapsto (0, \infty)$ be a continuous function satisfying (3.11), and let $\nu \in [0, 1]$. Then for every $f \in \mathcal{A}^+(\nu)$ there exist a sequence of polynomials $(Q_m)_{m=1}^\infty \subset \mathcal{A}^+(\nu)$, and a sequence $(\theta_m)_{m=1}^\infty$ decreasing to zero such that*

$$\lim_{m \rightarrow \infty} \|f - Q_m\|_{\mathcal{A}(\nu)} = 0$$

and

$$\sup \left\{ \frac{|1 - Q_m(\theta)|}{\varphi(\theta_m)\theta_m^{2-\nu}} : |\theta - \theta_m| \leq \varphi^{3/2}(\theta_m)\theta_m^{3-2\nu} \right\} \leq 10.$$

4. MAIN RESULTS

Using our construction of exponential polynomials from the previous section, we now produce a dense of functions F “almost” satisfying Erdős-Feller-Pollard’s condition (1.3) but having such a strong singularity at 1 that $(1-z)(1-F)^{-1}$ is unbounded in \mathbb{D} . To this aim, we employ either Baire category arguments (as in Theorem 4.2) or, alternatively, an iterative procedure (as in Theorem 4.4). Thus, we show that the Wiener type condition (1.3) is the best one can hope for as far as the boundedness of $(1-z)(1-F)^{-1}$ is concerned. We then use power weights $w = (k^\nu)_{k=1}^\infty, \nu \in [0, 1]$, to construct examples of $F \in \mathcal{A}^+(\nu)$ close to the constant function 1 on a sufficiently large set (but violating (1.3)). This will be used in the next section to study the property $(1-z)(1-F)^{-1} \in L^p$ for a fixed $p \in (1, \infty)$ in terms of the Taylor coefficients of F .

Proposition 4.1. *Let $L_\theta : X \mapsto (0, \infty), \theta \in (0, 1]$, be a family of continuous functionals on a complete metric space (X, ρ) , and let $c \geq 0$. Suppose that for any $f \in X$ there exists a sequence $(f_m)_{m=1}^\infty \subset X$ satisfying*

$$(4.1) \quad \lim_{m \rightarrow \infty} \rho(f, f_m) = 0 \quad \text{and} \quad \limsup_{m \rightarrow \infty} \inf_{\theta \in (0, 1/m]} L_\theta[f_m] \leq c.$$

Then there exists a residual set $S \subset X$ (in particular, dense in X) such that

$$(4.2) \quad \sup_{m \in \mathbb{N}} \inf_{\theta \in (0, 1/m]} L_\theta[f] \leq c, \quad f \in S,$$

Proof. Define a functional $\mathcal{F}_m, m \in \mathbb{N}$, on X by

$$\mathcal{F}_m(f) := \inf_{\theta \in (0, 1/m]} L_\theta[f], \quad f \in X.$$

Note that

$$(4.3) \quad \mathcal{F}_m(f) \leq \mathcal{F}_n(f), \quad f \in X, \quad n \geq m.$$

Since $(L_\theta)_{\theta \in (0, 1]}$ are continuous, \mathcal{F}_m is upper-semicontinuous on X for each $m \in \mathbb{N}$, by a standard argument. By e.g. [15, Theorem 9.17.3] for every $m \in \mathbb{N}$ the set S_m of continuity points of \mathcal{F}_m is residual. Hence

$$S := \cap_{m \in \mathbb{N}} S_m$$

is residual as well. Thus, by (4.1) and (4.3), for every $f \in S$ and every $m \in \mathbb{N}$:

$$\mathcal{F}_m(f) = \lim_{n \rightarrow \infty} \mathcal{F}_m(f_n) \leq \limsup_{n \rightarrow \infty} \mathcal{F}_n(f_n) \leq c,$$

and the statement follows. \square

Theorem 4.2. *Let $\nu \in [0, 1]$, and let $\varphi : (0, 1] \mapsto (0, \infty)$ be a continuous function satisfying (3.11). Then there exists a residual set $S \subset \mathcal{A}^+(\nu)$ of aperiodic functions with the following property: for every $f \in S$ there is a sequence $(\theta_m)_{m=1}^\infty \subset (0, 1]$ decreasing to zero such that*

$$(4.4) \quad |1 - f(\theta_m)| \leq 10\varphi(\theta_m)\theta_m^{2-\nu}, \quad |\theta - \theta_m| \leq \varphi^{3/2}(\theta_m)\theta_m^{3-2\nu}, \quad m \in \mathbb{N}.$$

Proof. Consider a family $(L_{\theta_0})_{\theta_0 \in (0,1]}$ of continuous functionals on $\mathcal{A}^+(\nu)$ given by

$$(4.5) \quad L_{\theta_0}[f] := \sup \left\{ \frac{|1 - f(\theta)|}{\varphi(\theta_0)\theta_0^{2-\nu}} : |\theta - \theta_0| \leq \varphi^{3/2}(\theta_m)\theta_m^{3-2\nu} \right\},$$

for each $\theta_0 \in (0,1]$. Using (2.1) and Corollary 3.9, we infer that $(L_{\theta})_{\theta \in (0,1]}$ satisfies the assumptions of Proposition 4.1 with $c = 10$. Therefore, there exists a residual set in $\mathcal{A}^+(\nu)$ satisfying (4.2) with $c = 10$. Note that the set of aperiodic functions in $\mathcal{A}^+(\nu)$ is residual. Taking the intersection of the two sets, we obtain a residual set satisfying (4.2) with $c = 10$ again. \square

Similarly, Corollary 3.5 and Proposition 4.1 imply the following statement.

Theorem 4.3. *Let $(\epsilon_k)_{k=1}^\infty$ be a positive sequence such that*

$$\lim_{k \rightarrow \infty} \epsilon_k = 0 \quad \text{and} \quad k\epsilon_k \geq 1, \quad k \in \mathbb{N}.$$

If $\tilde{w} = (k\epsilon_k)_{k=1}^\infty$ and $f \in \mathcal{A}^+(\tilde{w})$, then for each $\epsilon > 0$ there exists an aperiodic function $F \in \mathcal{A}^+(\tilde{w})$ such that

$$(4.6) \quad \|f - F\|_{\mathcal{A}(\tilde{w})} \leq \epsilon \quad \text{and} \quad \lim_{m \rightarrow \infty} \frac{|1 - F(\theta_m)|}{\theta_m} = 0$$

for some sequence $(\theta_m)_{m=1}^\infty$ decreasing to zero.

Now we present another approach to Theorem 4.2 avoiding the category arguments. Although the approach leads to a slightly weaker statement, it seems to be a bit more transparent.

Theorem 4.4. *Let $\nu \in [0,1)$ be fixed, and let φ satisfy (3.11). If $f \in \mathcal{A}^+(\nu)$ then for all $\epsilon > 0$ and $\delta > 0$ there exist an aperiodic function $F \in \mathcal{A}^+(\nu)$ and a sequence $(\theta_m)_{m=1}^\infty \subset (0,1]$ decreasing to zero such that $\|f - F\|_{\mathcal{A}(\nu)} \leq \epsilon$ and*

$$(4.7) \quad |1 - F(\theta)| \leq (10 + \delta)\varphi(\theta_m)\theta_m^{2-\nu} \quad \text{if} \quad |\theta - \theta_m| \leq \varphi^{3/2}(\theta_m)\theta_m^{3-2\nu}.$$

Proof. Let $\epsilon > 0$ and $\delta > 0$ be fixed. Without loss of generality we may assume that $f = P_0$ is an aperiodic polynomial and that $\epsilon \in (0,1)$ is so small that the ball

$$\{g \in \mathcal{A}^+(\nu) : \|f - g\|_{\mathcal{A}(\nu)} \leq \epsilon\}$$

consists of aperiodic functions.

We construct the sequences

$$(\theta_m)_{m=1}^\infty \subset (0,1) \quad \text{and} \quad (\delta_m)_{m=1}^\infty \subset (0,1)$$

with

$$(4.8) \quad \sum_{m=1}^\infty \delta_m \leq \epsilon,$$

and the sequence of polynomials $(P_m)_{m=1}^\infty \subset \mathcal{A}^+$ in such a way that for $1 \leq m \leq N$ and θ satisfying $|\theta - \theta_m| \leq \varphi^{3/2}(\theta_m)\theta_m^{3-2\nu}$ one has

$$(4.9) \quad |1 - P_N(\theta)| \leq \left(10 + \delta \sum_{j=1}^{N-1} \frac{1}{2^j}\right) \varphi(\theta_m)\theta_m^{2-\nu},$$

and moreover

$$(4.10) \quad \|P_{m-1} - P_m\|_{\mathcal{A}(\nu)} \leq \delta_m.$$

Let

$$\delta_1 = \min\{\delta/2, \epsilon/2\}.$$

By Corollary 3.8 there exists (large enough) $\theta_1 \in (0, 1)$ and a polynomial $P_1 \in \mathcal{A}^+$ such that

$$|1 - P_1(\theta)| \leq 10\varphi(\theta_1)\theta_1^{2-\nu},$$

whenever $|\theta - \theta_1| \leq \varphi^{3/2}(\theta_1)\theta_1^{3-2\nu}$, and

$$\|P_0 - P_1\|_{\mathcal{A}(\nu)} \leq \frac{4\pi}{\varphi^{1/2}(\theta_1)} \leq \delta_1.$$

So, (4.9) and (4.10) hold for $N = 1$.

Arguing by induction, suppose that (4.9) and (4.10) are true for some $N \geq 1$. Choose δ_{N+1} satisfying

$$\delta_{N+1} \leq \frac{\delta}{2^{N+1}} \min_{1 \leq m \leq N} (\varphi(\theta_m)\theta_m^{2-\nu}) \quad \text{and} \quad \delta_{N+1} \leq \frac{\epsilon}{2^{N+1}}.$$

Then by Corollary 3.8 applied to $P = P_N$ there exist $P_{N+1} \in \mathcal{A}^+$, and (small enough) $\theta_{N+1} < \theta_N$ such that

$$(4.11) \quad |1 - P_{N+1}(\theta)| \leq 10\varphi(\theta_{N+1})\theta_{N+1}^{2-\nu}$$

if $|\theta - \theta_{N+1}| \leq \varphi^{3/2}(\theta_{N+1})\theta_{N+1}^{3-2\nu}$, and, moreover,

$$\|P_N - P_{N+1}\|_{\mathcal{A}(\nu)} \leq \frac{4\pi}{\varphi^{1/2}(\theta_{N+1})} \leq \delta_{N+1}.$$

Hence for every $m \in [1, N]$ and every θ satisfying $|\theta - \theta_m| \leq \varphi^{3/2}(\theta_m)\theta_m^{3-2\nu}$ we have

$$\begin{aligned} |1 - P_{N+1}(\theta)| &\leq |1 - P_N(\theta)| + \|P_N - P_{N+1}\|_{\mathcal{A}(\nu)} \\ &\leq \left(10 + \delta \sum_{j=1}^{N-1} \frac{1}{2^j}\right) \varphi(\theta_m)\theta_m^{2-\nu} + \delta_{N+1} \\ &\leq \left(10 + \delta \sum_{j=1}^{N-1} \frac{1}{2^j}\right) \varphi(\theta_m)\theta_m^{2-\nu} + \delta \frac{\varphi(\theta_m)\theta_m^{2-\nu}}{2^{N+1}} \\ &\leq \left(10 + \delta \sum_{j=1}^N \frac{1}{2^j}\right) \varphi(\theta_m)\theta_m^{2-\nu}. \end{aligned}$$

Taking in account (4.11), we infer that (4.9) and (4.10) hold for $N + 1$ as well.

By (4.8) the sequence $(P_m)_{m=0}^\infty$ is Cauchy in $\mathcal{A}^+(\nu)$, therefore there exists $F \in \mathcal{A}^+(\nu)$ such that

$$\lim_{m \rightarrow \infty} \|F - P_m\|_{\mathcal{A}(\nu)} = 0 \quad \text{and} \quad \|P_0 - F\|_{\mathcal{A}(\nu)} \leq \sum_{m=1}^{\infty} \delta_m \leq \epsilon.$$

Finally, (4.9) yields (4.7), and moreover F is aperiodic by the above. \square

5. REGULARITY OF RECIPROCAL SERIES IN TERMS OF THE L^p -SCALE

In this section we will study the regularity of generating functions for renewal sequences with respect to the L^p -scale. Namely, we will be concerned with the identifying p such that $\frac{1-z}{1-F} \in L^p(\mathbb{T})$, where $F(z) = \sum_{k=1}^{\infty} a_k z^k$, $z \in \mathbb{T}$, with

$$a_k \geq 0, \quad \sum_{k=1}^{\infty} a_k = 1.$$

It is clear that it is enough to study the same issue for the function R_f defined by

$$R_f(\theta) = \frac{\theta}{1 - f(\theta)}, \quad \theta \in [-\pi, \pi].$$

where $f(\theta) = F(e^{i\theta})$. We will show, in particular, that while R_f has certain amount of regularity being in L^p for $p \in [1, 2]$, R_f is not very regular in the sense that R_f does not belong, in general, to L^p , if $p > 3$. This result will be put below into a more general (and sharper) context of the spaces $\mathcal{A}^+(\nu)$.

To get positive results on the regularity of R_f we will use an idea from [25]. In particular, we will use the following crucial result proved in [25, Theorem 1]. (The result formulated in [25] in a weaker form but the proof given there yields the statement given below.)

Theorem 5.1. *Let*

$$\sum_{k=1}^{\infty} a_k = 1, \quad a_k \geq 0,$$

and

$$f(\theta) = \sum_{k=1}^{\infty} a_k \cos(\mu_k \theta + \alpha_k), \quad \theta \in [-\pi, \pi],$$

with $\mu_k \geq 1$ and $\alpha_k \in \mathbb{R}$, $k \geq 1$. Then

$$(5.1) \quad \text{meas}(\{\theta \in [-\pi, \pi] : f(\theta) \geq 1 - \epsilon\}) \leq 4\pi\epsilon^{1/2}, \quad \epsilon \in (0, 1].$$

Recall that here and in the sequel meas stands for the Lebesgue measure.

Remark 5.2. Note that the above estimate is the best possible as the example of $f(\theta) = \cos \theta$ shows. Remark also that Theorem 5.1 was stated in [25] with a constant A instead of 4π above. The uniformity of A was not clarified in [25]. Since that property is crucial for our reasoning and to be on a safe side

we address we provide an independent proof of Theorem 5.1 in Appendix A.

Corollary 5.3. *Let $(a_k)_{k \geq m} \subset [0, \infty)$, $m \in \mathbb{N}$, and let $r = \sum_{k=m}^{\infty} a_k \in (0, 1]$. If*

$$f(\theta) = \sum_{k=m}^{\infty} a_k \cos k\theta,$$

then for every $\theta_0 \in (0, 1]$ such that $m\theta_0 \leq 1$, one has

$$(5.2) \quad \text{meas} (\{\theta \in [\theta_0, 2\theta_0] : r - f(\theta) \leq \epsilon\}) \leq 4\pi\theta_0 \sqrt{\frac{\epsilon}{r}}, \quad \epsilon \in (0, r].$$

Proof. If $\tilde{f}(t) = r^{-1}f(\theta_0(t+1))$ and $\epsilon \in (0, r]$ then by (5.1) we obtain:

$$\begin{aligned} & \text{meas} (\{\theta \in [\theta_0, 2\theta_0] : r - f(\theta) \leq \epsilon\}) \\ &= \theta_0 \text{meas} \left(\{t \in [0, 1] : \tilde{f}(t) \geq 1 - \epsilon/r\} \right) \\ &\leq 4\pi\theta_0 \sqrt{\frac{\epsilon}{r}}. \end{aligned}$$

□

We will also need the next technical estimate for distribution functions. Its proof is postponed to Appendix A.

Proposition 5.4. *Let $\Omega \subset [0, \infty)$ be a measurable set of finite measure, and let $\varphi : \Omega \mapsto (0, \infty)$ be a measurable function. Suppose that there are constants $r > \eta > 0$ and $A > 0$ such that $\eta \leq \varphi \leq r$ a.e. on Ω and, moreover,*

$$\text{meas} (\{s \in \Omega : \varphi(s) \leq t\}) \leq A \sqrt{\frac{t}{r}}, \quad t \in [\eta, r].$$

Then for all $d \geq 0$ and $p \geq 1$,

$$(5.3) \quad \int_{\Omega} \frac{ds}{(\varphi(s) + d)^p} \leq \frac{\text{meas}(\Omega)}{(r + d)^p} + \frac{2Ap}{\eta^{p-1/2} \sqrt{r} + d^p}.$$

Now we are ready to prove one of the main results of this section. It is an extension and sharpening of Littlewood's Theorem 2 from [25]. For

$$(5.4) \quad f(\theta) = \sum_{k=1}^{\infty} a_k e^{ik\theta}, \quad \theta \in [-\pi, \pi],$$

from \mathcal{A}^+ , define for $\theta \in (0, 1]$:

$$\begin{aligned} r(\theta) &:= \sum_{k > 1/\theta} a_k, \\ W(\theta) &:= \sum_{1 \leq k \leq 1/\theta} k a_k, \\ U(\theta) &:= \sum_{1 \leq k \leq 1/\theta} k^2 a_k. \end{aligned}$$

Note that

$$(5.5) \quad \theta_0 U(\theta_0) \leq W(\theta_0).$$

Theorem 5.5. *Let $f \in \mathcal{A}^+$ be given by (5.4) and let $\theta_0 \in (0, 1]$. Then for each $p \geq 1$ there exists $C_p > 0$ such that for every $\theta_0 \in (0, 1]$ with $r(\theta_0) < 1$:*

$$(5.6) \quad \int_{\theta_0}^{2\theta_0} |R_f(\theta)|^p d\theta \leq C_p \left\{ \frac{\theta_0}{W^p(\theta_0)} + \frac{\theta_0^{2-p} r^{p-1}(\theta_0)}{W^{2p-1}(\theta_0) + r^{p-1}(\theta_0) \theta_0 U^p(\theta_0)} \right\}.$$

Proof. Let $n = n(\theta_0)$ satisfy $n \leq 1/\theta_0 < n+1$, and let $p \geq 1$ be fixed. By assumption,

$$(5.7) \quad W(\theta_0) \geq \sum_{k=1}^n a_k = 1 - r(\theta_0) > 0 \quad \text{and} \quad U(\theta_0) \geq \sum_{k=1}^n a_k > 0.$$

If $\theta \in [\theta_0, 2\theta_0]$ and $1 \leq k \leq n$ then $0 < k\theta \leq 2n\theta_0 < 3\pi/4$ so that $\sin k\theta \geq k\theta/4$ and $\sin k\theta/2 \geq k\theta/\pi$. Hence for $\theta \in [\theta_0, 2\theta_0]$ we have

$$(5.8) \quad \sum_{k=1}^n a_k \sin k\theta \geq \frac{\theta_0}{4} \sum_{k=1}^n k a_k \geq \frac{\theta_0}{4} W(\theta_0),$$

and

$$(5.9) \quad \begin{aligned} \sum_{k=1}^n a_k (1 - \cos k\theta) &= 2 \sum_{k=1}^n a_k \sin^2(k\theta/2) \\ &\geq \frac{2\theta_0^2}{\pi^2} \sum_{k=1}^n k^2 a_k \geq \frac{2\theta_0^2}{\pi^2} U(\theta_0). \end{aligned}$$

If $r(\theta_0) = 0$, then by (5.8) we have

$$\int_{\theta_0}^{2\theta_0} |R_f(\theta)|^p d\theta \leq \int_{\theta_0}^{2\theta_0} \frac{\theta^p d\theta}{(\theta_0/4 W(\theta_0))^p} \leq 8^p \frac{\theta_0}{W^p(\theta_0)},$$

and (5.6) holds.

Let now $r(\theta_0) > 0$. For $y \in [0, r(\theta_0)]$ let $S(y)$ be the set of such $\theta \in [\theta_0, 2\theta_0]$ that

$$(5.10) \quad \varphi(\theta) := \sum_{k=n+1}^{\infty} a_k (1 - \cos k\theta) \leq y.$$

Corollary 5.3 then yields

$$(5.11) \quad |S(y)| \leq 4\pi\theta_0 \frac{\sqrt{y}}{\sqrt{r(\theta_0)}}, \quad y \in [0, r(\theta_0)].$$

Now, since

$$\sin^2 k\theta = 1 - \cos^2 k\theta \leq 2(1 - \cos k\theta),$$

by (5.10) again, we have

$$\begin{aligned} \left| \sum_{k=n+1}^{\infty} a_k \sin k\theta \right| &\leq \left(\sum_{k=n+1}^{\infty} a_k \right)^{1/2} \left(\sum_{k=n+1}^{\infty} a_k \sin^2 k\theta \right)^{1/2} \\ &\leq \sqrt{2}r^{1/2}(\theta_0)\varphi^{1/2}(\theta) \\ &\leq \sqrt{2r(\theta_0)y}, \quad \theta \in S(y). \end{aligned}$$

So, putting

$$\eta := \min \left\{ r(\theta_0), \frac{\theta_0^2 W^2(\theta_0)}{128r(\theta_0)} \right\},$$

we infer that for every $\theta \in S(\eta)$:

$$\begin{aligned} (5.12) \quad |\operatorname{Im} f(\theta)| &\geq \left| \sum_{k=1}^n a_k \sin k\theta \right| - \left| \sum_{k=n+1}^{\infty} a_k \sin k\theta \right| \\ &\geq \frac{\theta_0 W(\theta_0)}{4} - \sqrt{2r(\theta_0)\eta} \\ &\geq \frac{\theta_0 W(\theta_0)}{8}. \end{aligned}$$

We now estimate the left-hand side of (5.6) as follows. Write

$$\int_{\theta_0}^{2\theta_0} |R_f(\theta)|^p d\theta \leq J_{p,1}(\theta_0) + J_{p,2}(\theta_0),$$

where

$$\begin{aligned} J_{p,1}(\theta_0) &:= 2^p \theta_0^p \int_{\theta \in S(\eta)} \frac{d\theta}{|\operatorname{Im} f(\theta)|^p}, \\ J_{p,2}(\theta_0) &:= 2^p \theta_0^p \int_{\theta \in [\theta_0, 2\theta_0] \setminus S(\eta)} \frac{d\theta}{|1 - \operatorname{Re} f(\theta)|^p}. \end{aligned}$$

We deal with each of the terms $J_{p,1}$ and $J_{p,2}$ above separately. First, observe that by (5.12),

$$(5.13) \quad J_{p,1}(\theta_0) \leq 2^{4p} \frac{\theta_0}{W^p(\theta_0)}.$$

Second, if in addition $\eta \leq r(\theta_0)$, then

$$\frac{\theta_0}{r(\theta_0)} \leq \frac{8\sqrt{2}}{W(\theta_0)}.$$

Setting to simplify the notation

$$r = r(\theta_0), \quad W = W(\theta_0), \quad U = U(\theta_0), \quad d = \frac{2\theta_0^2 U(\theta_0)}{\pi^2},$$

and using Proposition 5.4 and (5.11), we obtain that

$$\begin{aligned} \frac{J_{p,2}(\theta_0)}{2^p} &\leq \theta_0^p \int_{\theta \in [\theta_0, 2\theta_0] \setminus S(\eta)} \frac{d\theta}{(\varphi(\theta) + d)^p} \\ &\leq \frac{\theta_0^{p+1}}{(r+d)^p} + \frac{8\pi p \theta_0^{p+1}}{\eta^{p-1/2} \sqrt{r} + d^p} \\ &\leq \frac{\theta_0^{p+1}}{r^p} + \frac{8\pi p \theta_0^{p+1}}{(\theta_0 W)^{2p-1} \sqrt{r}/(128r)^{p-1/2} + d^p} \\ &\leq \frac{(8\sqrt{2})^p \theta_0}{W^p} \\ &\quad + \frac{8\pi p \theta_0^{p+1}}{\theta_0^{2p-1} W^{2p-1} \sqrt{r}/(128r)^{p-1/2} + \theta_0^{2p} U^p/(2/\pi^2)^p} \\ &\leq \frac{(8\sqrt{2})^p \theta_0}{W^p} + \frac{8\pi p (128)^{p-1/2}}{\theta_0^{p-2} (W^{2p-1}/r^{p-1} + \theta_0 U^p)}. \end{aligned}$$

Taking in account (5.13) we infer that (5.6) holds with $C_p = 3p \cdot 2^{8p}$. \square

Theorem 5.5 allows us to describe the integrability of $1/(1-f)$ in terms of the size of Fourier coefficients of $f \in \mathcal{A}^+$. We will need the next simple proposition on series with positive terms proved in Appendix A.

Proposition 5.6. *Let $(q_k)_{k \geq 2^{m-1}+1}, m \in \mathbb{N}$, be a positive decreasing sequence, and let $\alpha \geq 0$. Then*

$$\frac{1}{2^\alpha} \sum_{n=2^m}^{\infty} n^{\alpha-1} q_n \leq \sum_{n=m}^{\infty} 2^{\alpha n} q_{2^n} \leq 2^{1+\alpha} \sum_{n=2^{m-1}+1}^{\infty} n^{\alpha-1} q_n.$$

For $f \in \mathcal{A}$ define

$$r_n := r(1/n), \quad W_n := W(1/n), \quad U_n := U(1/n), \quad n \in \mathbb{N}.$$

Corollary 5.7. *Let $f \in \mathcal{A}^+$ be given by (5.4). Then there exist $c > 0$ and $m_0 \in \mathbb{N}$ such that*

$$(5.14) \quad c^{-1} \sum_{n=m_0}^{\infty} \frac{1}{nW_n + n^2 r_n} \leq \int_0^{1/m_0} \frac{d\theta}{|1-f(\theta)|} \leq c \sum_{n=m_0}^{\infty} \frac{1}{nW_n}.$$

In particular, if f is aperiodic and the right-hand side of (5.14) is finite, then $1/(1-f) \in L^1[-\pi, \pi]$.

Remark 5.8. Observe that if $1/(1-f) \in L^1[-\pi, \pi]$ and $1/(1-F(z)) = \sum_{k=0}^{\infty} b_k z^k$, then $1/(1-F)$ belongs to the Hardy space $H^1(\mathbb{D})$, and by Hardy's

inequality

$$\sum_{n=1}^{\infty} n^{-1} b_n < \infty.$$

Proof. Choose $\Theta_0 \in (0, 1]$ with $r(\Theta_0) < 1$. By Theorem 5.5,

$$(5.15) \quad \int_{\theta_0}^{2\theta_0} \frac{d\theta}{|1-f(\theta)|} \leq \theta_0^{-1} \int_{\theta_0}^{2\theta_0} |R_f(\theta)| d\theta \leq \frac{c}{W(\theta_0)}, \quad \theta_0 \in (0, \Theta_0],$$

for a constant $c > 0$. Fix $n_0 \in \mathbb{N}$ such that

$$m_0 := 2^{n_0-1} \geq 1/\Theta_0,$$

and note that $W_n > 0, n \geq m_0$, and moreover $(W_n)_{n=m_0}^{\infty}$ monotonically increases. Using (5.15) and Proposition 5.6, we obtain:

$$\begin{aligned} \int_0^{1/m_0} \frac{d\theta}{|1-f(\theta)|} &= \sum_{n=n_0}^{\infty} \int_{1/2^n}^{1/2^{n-1}} \frac{d\theta}{|1-f(\theta)|} \\ &\leq c \sum_{n=n_0}^{\infty} \frac{1}{W_{2^n}} \leq 2c \sum_{n=2^{n_0-1}+1}^{\infty} \frac{1}{nW_n} \leq 2c \sum_{n=m_0}^{\infty} \frac{1}{nW_n}, \end{aligned}$$

that is the right-hand side estimate in (5.14) holds. If the series $\sum_{n=m_0}^{\infty} \frac{1}{nW_n}$ converges, then since f is aperiodic and $|R_f(\theta)|$ is symmetric in the sense that $|R_f(\theta)| = |R_f(-\theta)|$, we infer that $1/(1-f) \in L^1[-\pi, \pi]$.

To prove the left-hand side estimate in (5.14), we note that if $n \in \mathbb{N}$ and $\theta \in (0, 1]$ are such that $n \leq 1/\theta < n+1$, then, using (5.5), we have

$$\begin{aligned} (5.16) \quad |1-f(\theta)| &\leq \sum_{k=1}^{\infty} a_k (1 - \cos k\theta) + \left| \sum_{k=1}^{\infty} a_k \sin k\theta \right| \\ &\leq \frac{\theta_0^2}{2} U_n + \theta_0 W_n + 2r_n \\ &\leq 2(W_n/n + r_n). \end{aligned}$$

Therefore, from (5.16) it follows that there exists $m_0 \in \mathbb{N}$ such that

$$\begin{aligned} \int_0^{1/m_0} \frac{d\theta}{|1-f(\theta)|} &= \sum_{n=m_0}^{\infty} \int_{1/(n+1)}^{1/n} \frac{d\theta}{|1-f(\theta)|} \geq \frac{1}{2} \sum_{n=m_0}^{\infty} \frac{1}{W_n/n + r_n} \int_{1/(n+1)}^{1/n} d\theta \\ &= \frac{1}{2} \sum_{n=m_0}^{\infty} \frac{1}{(W_n/n + r_n)n(n+1)} \geq \frac{1}{4} \sum_{n=m_0}^{\infty} \frac{1}{nW_n + n^2 r_n}. \end{aligned}$$

(Since we do not need Theorem 5.5 for the left-hand side estimate, a dyadic partition of $[0, 1/m_0]$ is replaced with a partition, in a sense, more convenient for writing down the final estimate.) \square

Remark 5.9. From (5.15) it follows that if

$$\mu = \sum_{k=1}^{\infty} k a_k = \infty,$$

then

$$\lim_{\theta_0 \rightarrow 0} \int_{\theta_0}^{2\theta_0} \frac{d\theta}{|1 - f(\theta)|} = 0,$$

cf. Littlewood's result (1.7). On the other hand, the left-hand side inequality in (5.14) shows that if $f \in \mathcal{A}^+$ is such that $\mu < \infty$ then

$$(5.17) \quad \int_0^\delta \frac{d\theta}{|1 - f(\theta)|} = \infty,$$

since in this case

$$W_n + nr_n = \sum_{k=1}^n ka_k + n \sum_{k=n+1}^{\infty} a_k \leq \mu, \quad n \in \mathbb{N},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{nW_n + n^2r_n} \geq \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

On the other hand, (5.17) is a direct consequence of $\lim_{\theta \rightarrow 0} (f(\theta) - 1)/\theta = \mu \in (0, \infty)$.

We pause now to illustrate Corollary 5.7 by the following example.

Example 5.10. Consider

$$f_\epsilon(\theta) := c_\epsilon \sum_{k=1}^{\infty} \frac{\log^\epsilon(k+1)}{k^2} e^{ik\theta}, \quad f_\epsilon(0) = 1, \quad \epsilon \geq 0.$$

Then $f \in \mathcal{A}^+$, and it is aperiodic. For each $\epsilon > 0$ we have

$$W_n = \sum_{k=1}^n \frac{\log^\epsilon(k+1)}{k} \geq c \log^{1+\epsilon}(n+1),$$

hence $1/(1 - f_\epsilon) \in L^1[-\pi, \pi]$ by (5.14). On the other hand, if $\epsilon = 0$ then

$$nW_n + n^2r_n \leq n \sum_{k=1}^n \frac{1}{k} + n^2 \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq cn \log(n+1),$$

for some constant $c > 0$, so (5.14) implies that $1/(1 - f_0) \notin L^1[-\pi, \pi]$. (Similarly, if $f \in \mathcal{A}^+$ is given by

$$f(\theta) := \sum_{k=1}^{\infty} \frac{e^{ik\theta}}{k(k+1)} = 1 + (1 - e^{-i\theta}) \log(1 - e^{i\theta}),$$

then f is aperiodic and $1/(1 - f) \notin L^1[-\pi, \pi]$.)

Although, in general, $R_f \notin L^p[-\pi, \pi]$, $p > 3$, it is possible to formulate a sufficient condition on the Fourier coefficients f ensuring $R_f \in L^p[-\pi, \pi]$ for fixed $p > 2$. (This way we may also produce $R_f \in \cap_{p>2} L^p[-\pi, \pi]$.) The next statement is a direct implication of Theorem 5.5.

Corollary 5.11. *Let $f \in \mathcal{A}^+$ be aperiodic, and let $p > 2$. If there is $m_0 \in \mathbb{N}$ such that*

$$(5.18) \quad \sum_{n=m_0}^{\infty} \frac{n^{p-2} r_n^{p-1}}{n W_n^{2p-1} + r_n^{p-1} U_n^p} < \infty,$$

then $R_f \in L^p[-\pi, \pi]$.

Proof. By aperiodicity f and symmetry of $|R_f(\theta)|$ it suffices to prove that

$$(5.19) \quad \int_0^\delta |R_f(\theta)|^p d\theta < \infty$$

for some $\delta > 0$. Using (5.18), Theorem 5.5 and Proposition 5.6, we infer that there exists $c > 0$ such that for large enough n_0 and $m_0 = 2^{n_0-1} + 1$:

$$\begin{aligned} \int_0^{1/2^{n_0-1}} |R_f(\theta)|^p d\theta &= \sum_{n=n_0}^{\infty} \int_{1/2^n}^{1/2^{n-1}} |R_f(\theta)|^p d\theta \\ &\leq c \sum_{n=1/2^{n_0}}^{\infty} \left[\frac{1}{2^n W_{2^n}^p} + \frac{2^{(p-1)n} r_{2^n}^{p-1}}{2^n W_{2^n}^{2p-1} + r_{2^n}^{p-1} U_{2^n}^p} \right] \\ &\leq c 2^p \sum_{n=m_0}^{\infty} \left[\frac{1}{n^2 W_n^p} + \frac{n^{p-2} r_n^{p-1}}{n W_n^{2p-1} + r_n^{p-1} U_n^p} \right] < \infty. \end{aligned}$$

□

Further we will make use of Theorem 5.5 to describe the regularity R_f with respect to the L^p -scale for arbitrary $f \in \mathcal{A}^+(\nu)$, $\nu > 0$.

Corollary 5.12. *Let $\nu \in (0, 1)$, and $f \in \mathcal{A}^+(\nu)$ be aperiodic. Then $R_f \in L^p[-\pi, \pi]$, where $p = 1 + \frac{1}{1-\nu}$. On the other hand, for any $\nu \in [0, 1)$ and*

$$p \in \left(2 + \frac{1}{1-\nu}, \infty \right)$$

there exists an aperiodic function $f \in \mathcal{A}^+(\nu)$ such that $R_f \notin L^p[-\pi, \pi]$.

Proof. Let us prove the first claim. Again, by aperiodicity of f and symmetry of $|R_f(\theta)|$ it suffices to show that (5.19) holds for some $\delta > 0$ and $p = 1 + 1/(1 - \nu)$. Choose $n_0 \in \mathbb{N}$ so large that $W_n \geq 1/2$, $n \geq n_0$. Then, taking into account [18, Thm. 165], we conclude that there are constants

$c > 0$ and $\tilde{c} > 0$ such that

$$\begin{aligned}
\sum_{n=n_0}^{\infty} \frac{n^{p-2} r_n^{p-1}}{n W_n^{2p-1} + r_n^{p-1} U_n^p} &\leq 2^{2p-1} \sum_{n=n_0}^{\infty} n^{p-3} \left(\sum_{k=n+1}^{\infty} a_k \right)^{p-1} \\
&\leq c \sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^n k^{p-3} \right)^{1/(p-1)} \\
&\leq \tilde{c} \sum_{n=1}^{\infty} a_n n^{(p-2)/(p-1)} \\
&= \tilde{c} \sum_{n=1}^{\infty} a_n n^{\nu} < \infty.
\end{aligned}$$

Thus, Corollary 5.11 implies that $R_f \in L^p[-\pi, \pi]$.

Second, to prove the negative result, let $p \in \left(2 + \frac{1}{1-\nu}, \infty\right)$ be fixed. Write

$$p = 2 + \frac{1}{1-\nu} + \delta,$$

and for a fixed $\epsilon \in (0, 1-\nu)$ define a continuous function

$$\varphi(\theta) := \theta^{-\epsilon}, \quad \theta \in (0, 1].$$

Since φ satisfies (3.11), Theorem 4.2 implies that there exists an aperiodic $f = f_{\epsilon} \in \mathcal{A}^+(\nu)$ and a decreasing sequence $(\theta_m)_{m=1}^{\infty} \subset (0, 1]$ tending to zero such that

$$|1 - f(\theta)| \leq c \theta_m^{2-\nu-\epsilon}$$

whenever $|\theta - \theta_m| \leq \theta_m^{3-2\nu}$. Setting

$$\beta = (1 - \nu - \epsilon)p - 3 + 2\nu$$

we have

$$(5.20) \quad \int_{\theta_m}^{\theta_m + \theta_m^{3-2\nu}} \frac{\theta^p d\theta}{|1 - f(\theta)|^p} \geq \frac{c}{\theta_m^{p(2-\nu-\epsilon)}} \int_{\theta_m}^{\theta_m + \theta_m^{3-2\nu}} \theta^p d\theta \geq \frac{c}{\theta_m^{\beta}}$$

for some constant $c > 0$. Note that

$$\beta = (1 - \nu)\delta - \epsilon(3 - 2\nu + \delta),$$

and choose ϵ such that $\beta > 0$, that is

$$0 < \epsilon < \frac{(1 - \nu)\delta}{3 - 2\nu + \delta}.$$

As $\theta_m \rightarrow 0$, $m \rightarrow \infty$, the right-hand side of (5.20) tends to infinity as $m \rightarrow \infty$, hence $R_f \notin L^p[-\pi, \pi]$. \square

Finally, as a consequence of Corollary 5.12, we derive a result on the regularity of R_f measured in terms of L^p -spaces. The result corresponds formally to the case $\nu = 0$ in Corollary 5.12 and should be compared to the property (1.5) discussed by Erdős, de Bruijn and Kingman. While its

positive part is elementary, it was apparently overlooked by specialists in probability theory.

Corollary 5.13. *Let $f \in \mathcal{A}^+$ be given by (5.4), and let $(b_k)_{k=0}^\infty$ be a renewal sequence associated to $(a_k)_{k=1}^\infty$. Then $R_f \in L^2[-\pi, \pi]$ and*

$$(5.21) \quad \sum_{k=0}^{\infty} (b_{k+1} - b_k)^2 < \infty.$$

At the same time, there exists an aperiodic $f \in \mathcal{A}^+$ such that $R_f \notin L^p[-\pi, \pi]$ for every $p \in (3, \infty]$.

Proof. The second claim follows directly from Corollary 5.12. To prove the first claim recall that $f(\theta) = F(e^{i\theta})$ where $F(z) = \sum_{k=1}^{\infty} a_k z^k$, $z \in \mathbb{D}$. Note that $\operatorname{Re}(1/(1-F)) \in L^1(\mathbb{T})$ since

$$\int_{-\pi}^{\pi} \operatorname{Re} \frac{1}{1-F(e^{i\theta})} d\theta \leq \limsup_{r \rightarrow 1} \int_{-\pi}^{\pi} \operatorname{Re} \frac{1}{1-F(re^{i\theta})} d\theta = 2\pi$$

by Fatou's Lemma and positivity of the harmonic function $(r, \theta) \rightarrow \operatorname{Re}(1 - F(e^{i\theta}))^{-1}$ in \mathbb{D} , cf. [21, p. 10-12]. On the other hand, if n_0 is such that $a_{n_0} \neq 0$ and $\theta \in [-1/n_0, 1/n_0]$, then

$$\operatorname{Re}(1 - F(e^{i\theta})) \geq a_{n_0}(1 - \cos n_0\theta) \geq \frac{2\theta^2}{\pi^2} n_0^2 a_{n_0}$$

and

$$\operatorname{Re} \frac{1}{1-F(e^{i\theta})} = \frac{\operatorname{Re}(1 - F(e^{i\theta}))}{|1 - F(e^{i\theta})|^2} \geq 2a_{n_0} \left(\frac{n_0}{\pi}\right)^2 (R_f(\theta))^2.$$

Thus $R_f \in L^2[-1/n_0, 1/n_0]$, and, if f is aperiodic, then $R_f \in L^2[-\pi, \pi]$. As the latter property is equivalent to $(1-z)(1-F)^{-1} \in L^2(\mathbb{T})$, Parseval's identity yields (5.21). \square

Remark 5.14. For $f \in \mathcal{A}^+$ and $G_\epsilon := \{\theta \in [-\pi, \pi] : \operatorname{Re} f(\theta) \geq 1 - \epsilon\}$ there are several estimates in the literature of the form

$$\operatorname{meas}(G_\epsilon) \leq C\sqrt{\epsilon}$$

for an absolute constant $C > 0$, see e.g. [4], [9], [10], [23], [28]. The estimates are motivated by applications in probability theory and number theory, and they seem to be weaker than the estimate (5.1) provided by Theorem 5.1. (A related bound for f in terms of its coefficients has been given in [5].) To clarify their relations to our treatment, for $\epsilon \in (0, 1]$ and $f \in \mathcal{A}^+$ define

$$E_\epsilon := \{\theta \in [-\pi, \pi] : |1 - f(\theta)| \leq \epsilon\}.$$

Note that

$$\operatorname{meas}(E_\epsilon) \leq \operatorname{meas}(G_\epsilon) \leq C\sqrt{\epsilon}.$$

On the other hand, if f is aperiodic then Littlewood's theorem (1.8) implies that for every $\alpha \in (0, 1)$ there exists $c_\alpha(f) > 0$ such that

$$(5.22) \quad \operatorname{meas}(E_\epsilon) \leq c_\alpha(f)\epsilon^\alpha.$$

Indeed, let $\alpha \in (0, 1)$ be fixed. Then, taking $\gamma \in (0, 1 - \alpha)$ and using (1.8), we have

$$\begin{aligned} \epsilon^{-\alpha} \text{meas}(E_\epsilon) &\leq \int_{E_\epsilon} \frac{d\theta}{|1 - f(\theta)|^\alpha} \leq \int_{-\pi}^{\pi} \frac{d\theta}{|1 - f(\theta)|^\alpha} \\ &\leq \left(\int_{-\pi}^{\pi} \frac{\theta^{\gamma/\alpha} d\theta}{|1 - f(\theta)|} \right)^\alpha \left(\int_0^\pi \frac{d\theta}{\theta^{\gamma/(1-\alpha)}} \right)^{1-\alpha} := c_\alpha(f), \end{aligned}$$

and (5.22) follows. Thus (1.8) gives asymptotically better bounds for $\text{meas}(E_\epsilon)$. However it is not clear whether the constant $c_\alpha(f)$ can be taken independent of f .

6. REMARKS ON p -FUNCTIONS

The theory of p -functions can be considered as a discrete counterpart of the theory of renewal sequences. Its basic facts can be found in [21]. To put the relevant considerations on p -functions into our setting let us first recall a couple of basic facts.

Let a function φ in the right half-plane be defined as

$$(6.1) \quad \varphi(z) = z + c + \int_{(0, \infty)} (1 - e^{-zt}) \nu(dt),$$

where $c \geq 0$, and ν is a positive Borel measure on $(0, \infty)$ satisfying

$$\int_{(0, \infty)} \frac{t \nu(dt)}{1 + t} < \infty.$$

(Note that φ is a so-called Bernstein function, see [29] for more on that relevant function class.) Then there is a unique continuous function $g : [0, \infty) \mapsto [0, 1]$, $g(0) = 1$, called a (standard) p -function, such that

$$(6.2) \quad \int_0^\infty e^{-zt} g(t) dt = \frac{1}{\varphi(z)}, \quad z > 0.$$

for $z > 0$ and then for z with $\text{Re } z \geq 0$. We refer to [21, Chapter 3] concerning basic facts of the analytic theory of p -functions. For p -function g given by (6.2) and (6.1) we will write $g \sim (c, \nu)$.

Moreover, for $g \sim (c, \nu)$ one has

$$g(\infty) := \lim_{t \rightarrow \infty} g(t) = \begin{cases} \frac{1}{1 + \int_{(0, \infty)} t \nu(dt)} \in [0, 1], & \text{if } c = 0, \\ 0, & \text{if } c > 0. \end{cases}$$

It was proved in [20, Theorem 3] (see also [21, p. 75-76]) that if $g(\infty) > 0$, i.e. if

$$(6.3) \quad c = 0 \quad \text{and} \quad \int_{(0, \infty)} t \nu(dt) < \infty,$$

then g has bounded variation on $[0, \infty)$. As in the setting of renewal sequences, a natural question is whether g has always bounded variation, i.e. also in the case when $g(\infty) = 0$. The question was asked by J. Kingman in

[21, p. 76], and soon after J. Hawkes produced in [17] an example showing that the answer is “no” in general.

The argument in [17] was based on the following observation. Let

$$\varphi_0(z) := \int_0^\infty (1 - e^{-zt}) \nu(dt), \quad \int_{(0,\infty)} \nu(dt) < \infty, \quad \operatorname{Re} z \geq 0,$$

and the corresponding p -function g be given by

$$\int_0^\infty e^{-zt} g(t) dt = \frac{1}{z + \varphi_0(z)}.$$

If g has bounded variation on $[0, \infty)$ and $g(\infty) = 0$, then

$$(6.4) \quad \lim_{\theta \rightarrow 0} \frac{i\theta}{i\theta - \varphi_0(-i\theta)} = 0.$$

Essentially, Hawkes constructed a quasi-exponential series

$$f(\theta) = \sum_{k=1}^\infty a_k e^{i\lambda_k \theta}, \quad a_k \geq 0,$$

(see the introduction) such that

$$\sum_{k=1}^\infty a_k < \infty, \quad \sum_{k=1}^\infty k a_k = \infty \quad \text{and} \quad \lim_{\theta \rightarrow 0} \frac{|1 - f(\theta)|}{\theta} = 0$$

(thus $\lim_{\theta \rightarrow 0} \frac{|1 - f(\theta)|}{\theta}$ does not exist). Then, setting

$$\varphi_0(z) := 1 - \sum_{k=1}^\infty a_k e^{-\lambda_k z}$$

for z with $\operatorname{Re} z \geq 0$, one obtains the desired (counter-)example.

One can prove that in Hawkes' example

$$(6.5) \quad \sum_{k=1}^\infty k^\alpha a_k < \infty, \quad \alpha \in [0, 1/3].$$

In other words, the example states that there exists a finite (discrete) Borel measure ν on $(0, \infty)$, satisfying

$$\int_{(0,\infty)} t^\alpha \nu(dt) < \infty, \quad \alpha \in [0, 1/3],$$

such that the corresponding p -function $g \sim (0, \nu)$ has unbounded variation on $[0, \infty)$. If one can arrange $\operatorname{supp} \nu \subset \mathbb{N}$, then the example could also be used to produce a negative answer to the question by Erdős-de Bruijn-Kingman on renewal sequences. However, we do not see how to realize that in a way different to what was done above.

On the other hand, using our results, we can generalize the considerations by Hawkes in the following way thus showing that the condition (6.3) is the best possible in a sense (as (1.3), an analogue of (6.3), in the discrete setting).

Theorem 6.1. *Let $\beta : (0, \infty) \rightarrow (0, \infty)$ be a function satisfying*

$$\lim_{t \rightarrow \infty} \beta(t) = 0 \quad \text{and} \quad t\beta(t) \geq 1 \quad t \geq 1.$$

Then there exists a finite (discrete) Borel measure $\nu = \nu_\beta$ on $(0, \infty)$ such that

$$(6.6) \quad \int_{(0, \infty)} t\beta(t)\nu(dt) < \infty,$$

and the corresponding p -function $g \sim (0, \nu)$ has unbounded variation on $[0, \infty)$.

Proof. By Theorem 4.3, setting $w = (k\beta(k))_{k=1}^\infty$, there exists $f \in \mathcal{A}^+(w)$ such that

$$(6.7) \quad \lim_{\theta \rightarrow 0} \frac{|1 - f(\theta)|}{\theta} = 0.$$

If $F(e^{i\theta}) = f(\theta)$, then $z \rightarrow F(e^{-z})$ is of the form (6.1) for an appropriate discrete measure ν supported by \mathbb{N} . If a p -function g is defined by

$$\int_0^\infty e^{-zt}g(t)dt = \frac{1}{z + 1 - F(z)}, \quad \operatorname{Re} z > 0,$$

then g satisfies (6.6), and, moreover, it has unbounded variation on $[0, \infty)$ by (6.4) and (6.7). □

7. APPENDIX A: TECHNICALITIES

The proof of Lemma 3.1 Since $\lambda/2 \in (0, \pi/2)$, $\gamma/2 \in (0, \pi/2)$ and $(\gamma + \lambda)/2 \in (0, \pi/3)$, we have

$$\frac{\lambda}{4\gamma} \leq \frac{\lambda}{\pi\gamma} \leq d_{\lambda, \gamma} \leq \frac{\pi\lambda}{4\gamma} \leq \frac{\lambda}{\gamma}.$$

Moreover, taking into account that

$$|1 - e^{ia}|^2 = 4\sin^2(a/2)$$

and

$$\operatorname{Re}((1 - e^{ia})(1 - e^{ib})) = -4\sin(a/2)\sin(b/2)\cos((b+a)/2),$$

for $a, b \in \mathbb{R}$, we obtain

$$|(1 - e^{i\lambda}) + d(1 - e^{-i\gamma})|^2 = A_\gamma \left(d - \frac{B_{\lambda, \gamma}}{A_\gamma} \right)^2 + D_{\lambda, \gamma},$$

where

$$\begin{aligned} A_\gamma &:= 4 \sin^2(\gamma/2), \\ B_{\lambda,\gamma} &:= -\operatorname{Re}(1 - e^{i\lambda})(1 - e^{i\gamma}) \\ &= 4 \sin(\gamma/2) \sin(\lambda/2) \cos((\gamma + \lambda)/2), \\ C_\lambda &:= 4 \sin^2(\lambda/2), \\ D_{\lambda,\gamma} &:= C_\lambda - \frac{B_{\lambda,\gamma}^2}{A_\gamma}. \end{aligned}$$

Therefore,

$$\frac{B_{\lambda,\gamma}}{A_\gamma} = d_{\lambda,\gamma}, \quad D_{\gamma,\lambda} = 4 \sin^2(\lambda/2) \sin^2((\gamma + \lambda)/2),$$

and then

$$|(1 - e^{i\lambda}) + d_{\lambda,\gamma}(1 - e^{-i\gamma})| = 2 \sin(\lambda/2) \sin((\gamma + \lambda)/2) \leq \frac{\lambda(\gamma + \lambda)}{2}.$$

□

To prove Littlewood's result mentioned in Section 5 with an explicit constant we prove first the next auxiliary estimate, see also Remark 5.2.

Lemma 7.1. *If $E \subset [a, a + 2\pi]$ is a measurable set such that*

$$\int_E (1 - \cos t) dt \leq \epsilon$$

for some $\epsilon > 0$, then

$$(7.1) \quad \operatorname{meas}(E) \leq (4\pi^2)^{1/3} \epsilon^{1/3}.$$

Proof. By 2π -periodicity of $\cos t$, we may assume without loss of generality that $E \subset [-\pi, \pi]$. Write $I = [-\operatorname{meas}(E)/2, \operatorname{meas}(E)/2]$ and

$$E_1 = E \cap I, \quad E_2 = E \setminus E_1, \quad \text{and} \quad \tilde{E} = I \setminus E_1.$$

Noting that $\operatorname{meas}(\tilde{E}) = \operatorname{meas}(E_2)$, we have

$$\begin{aligned} \int_{E_2} (1 - \cos t) dt &\geq \operatorname{meas}(E_2) \min_{t \in E_2} (1 - \cos t) \\ &\geq \operatorname{meas}(\tilde{E}) \max_{t \in \tilde{E}} (1 - \cos t) \\ &\geq \int_{\tilde{E}} (1 - \cos t) dt, \end{aligned}$$

hence

$$\begin{aligned} \epsilon &\geq \int_E (1 - \cos t) dt \geq \int_{E_1} (1 - \cos t) dt + \int_{\tilde{E}} (1 - \cos t) dt \\ &= \int_I (1 - \cos t) dt, \end{aligned}$$

so that, taking into account that

$$t - \sin t \geq \frac{t^3}{\pi^2}, \quad t \in [0, \pi/2],$$

we obtain

$$\epsilon \geq \int_I (1 - \cos t) dt = 2(t - \sin t)|_{t=\text{meas}(E)/2} \geq \frac{1}{4\pi^2} (\text{meas}(E))^3,$$

and the statement follows. \square

Corollary 7.2. *Let $\alpha \geq 1$, $\beta \in \mathbb{R}$, $\epsilon > 0$, and let a measurable set $E \subset [a, a + 2\pi]$, $a \in \mathbb{R}$, be such that*

$$\int_E (1 - \cos(\alpha t + \beta)) dt \leq \epsilon.$$

Then

$$(7.2) \quad \text{meas}(E) \leq (16\pi^2)^{1/3} \epsilon^{1/3}.$$

Proof. We have

$$\int_E (1 - \cos(\alpha t + \beta)) dt = \frac{1}{\alpha} \int_G (1 - \cos t) dt,$$

where $G = \alpha E + \beta \subset [\alpha a + \beta, \alpha a + \beta + 2\pi\alpha]$ and $\text{meas}(G) = \alpha \text{meas}(E) \leq 2\pi\alpha$. Set $n = [\alpha + 1]$, $h = 2\pi\alpha/n$, and write

$$G_k = [h_k, h_k + h] \cap G, \quad h_k = \alpha a + \beta + (k-1)h, \quad k = 1, \dots, n,$$

so that

$$G = \cup_{k=1}^n G_k, \quad \text{where} \quad G_k \cap G_j = \emptyset, \quad k \neq j, \quad 1 \leq k \leq n.$$

By assumption,

$$\int_{G_k} (1 - \cos t) dt \leq \alpha \epsilon_k, \quad k = 1, \dots, n, \quad \epsilon_1 + \dots + \epsilon_n = \epsilon.$$

Hence, by Lemma 7.1 and convexity arguments,

$$\begin{aligned} \text{meas}(E) &= \alpha^{-1} \sum_{k=1}^n \text{meas}(G_k) \leq (4\pi^2)^{1/3} \alpha^{-2/3} \sum_{k=1}^n \epsilon_k^{1/3} \\ &\leq (4\pi^2)^{1/3} \left(\frac{n}{\alpha}\right)^{2/3} \epsilon^{1/3} \\ &\leq (16\pi^2)^{1/3} \epsilon^{1/3}. \end{aligned}$$

\square

Thus, we have the following result, formulated essentially in [25, Lemma] with a constant A instead of $(16\pi^2)^{1/3}$.

Lemma 7.3. *Let*

$$f(t) = \sum_{m=1}^{\infty} a_m \cos(\alpha_m t + \beta_m)$$

where $\alpha_m \geq 1$, $\beta_m \in \mathbb{R}$, $m \in \mathbb{N}$, and

$$a_m \geq 0, \quad \sum_{m=1}^{\infty} a_m = 1.$$

If $\epsilon > 0$ and $E \subset [-\pi, \pi]$ is a measurable set such that

$$\int_E (1 - f(t)) dt \leq \epsilon,$$

then $\text{meas}(E) \leq (16\pi^2)^{1/3} \epsilon^{1/3}$.

Now we are able to prove Theorem 5.1 with the absolute constant 4π .

Proof of Theorem 5.1. Let f be as in Theorem 5.1, and let $E_\epsilon := \{t \in [-\pi, \pi] : f(t) \geq 1 - \epsilon\}$. Then

$$\int_{E_\epsilon} (1 - f(t)) dt \leq \epsilon \text{meas}(E_\epsilon),$$

and from Lemma 7.3 it follows that $\text{meas}(E_\epsilon) \leq (16\pi^2)^{1/3} \epsilon^{1/3} (\text{meas}(E_\epsilon))^{1/3}$, hence

$$\text{meas}(E_\epsilon) \leq (16\pi^2)^{1/2} \epsilon^{1/2} = 4\pi \epsilon^{1/2}.$$

□

We now turn to the proof of Proposition 5.4. First, let us recall one of the forms of layer cake representation, which is a direct consequence of e.g. [14, Prop. 1.3.4] or [24, Ch. 1.13].

Lemma 7.4. *Let $\psi : [0, \infty) \mapsto (0, \infty)$ be a differentiable strictly decreasing function with $\lim_{t \rightarrow \infty} \psi(t) = 0$, and let $\varphi : \Omega \mapsto (0, \infty)$ be a measurable function, where $\Omega \subset [0, \infty)$ is a measurable set of finite measure. Then*

$$(7.3) \quad \int_{\Omega} \psi(\varphi(s)) ds = - \int_0^{\infty} \psi'(t) g_{\varphi}(t) dt,$$

where

$$(7.4) \quad g_{\varphi}(t) := \{s \in \Omega : \varphi(s) \leq t\}, \quad t > 0.$$

(It suffices to note that

$$\int_{\Omega} \psi(\varphi(s)) ds = - \int_{\Omega} [\psi(0) - \psi(\varphi(s))] ds + \psi(0)|\Omega|$$

and apply either of the statements from [14] or [24] to $\psi(0) - \psi(t)$.)

Having Lemma 7.4 in mind, we are now ready to give a proof of Proposition 5.4.

Proof of Proposition 5.4. Let g_{φ} be defined by (7.4) so that

$$g_{\varphi}(t) = \text{meas}(\Omega) - \text{meas}\{s : \varphi(s) > t\}, \quad t > 0.$$

By assumption,

$$g_\varphi(t) \leq A\sqrt{\frac{t}{r}}, \quad t \in [\eta, r].$$

Using Corollary 7.4 with $\psi(t) = 1/(t+d)^p$ we obtain

$$(7.5) \quad \int_{\Omega} \frac{ds}{(\varphi(s)+d)^p} = p \int_0^\infty \frac{g_\varphi(t) dt}{(t+d)^{p+1}} = \frac{\text{meas}(\Omega)}{(r+d)^p} + p \int_\eta^r \frac{g_\varphi(t) dt}{(t+d)^{p+1}}.$$

Next, using the estimate (see e.g. [18, Thm. 41])

$$\frac{1}{x^q} - \frac{1}{y^q} \leq \frac{q(y-x)}{x^q y}, \quad y \geq x > 0, \quad q \geq 1,$$

and the elementary inequality

$$(x+y)^q \geq x^q + y^q, \quad x, y \geq 0, \quad q \geq 1,$$

we have

$$\begin{aligned} \int_\eta^r \frac{g_\varphi(t) dt}{(t+d)^{p+1}} &\leq \frac{A}{\sqrt{r}} \int_\eta^r \frac{\sqrt{t} dt}{(t+d)^{p+1}} \\ &\leq \frac{A}{\sqrt{r}} \int_\eta^r \frac{dt}{(t+d)^{p+1/2}} \\ &= \frac{A}{(p-1/2)\sqrt{r}} \left[\frac{1}{(\eta+d)^{p-1/2}} - \frac{1}{(r+d)^{p-1/2}} \right] \\ &\leq \frac{A}{(p-1/2)\sqrt{r}} \frac{(p-1/2)(r-\eta)}{(\eta+d)^{p-1/2}(r+d)} \\ &\leq \frac{A\sqrt{r}}{(\eta+d)^{p-1/2}(r+d)} \\ &\leq \frac{2A}{(\eta+d)^{p-1/2}(\sqrt{r}+\sqrt{d})} \\ &\leq \frac{2A}{\eta^{p-1/2}\sqrt{r}+d^p}. \end{aligned}$$

From this and (7.5) we obtain (5.3). \square

We finish this section with the proof of auxiliary Proposition 5.6 on positive series.

Proof of Proposition 5.6. Note that

$$2^{\alpha n} q_{2^n} \leq 2^\alpha j^\alpha q_j, \quad 2^{n-1} + 1 \leq j \leq 2^n, \quad n \geq m,$$

so

$$2^{\alpha n} q_{2^n} \leq \frac{2^\alpha}{2^{n-1}} \sum_{j=2^{n-1}+1}^{2^n} j^\alpha q_j \leq 2^{1+\alpha} \sum_{j=2^{n-1}+1}^{2^n} j^{\alpha-1} q_j,$$

and then

$$\sum_{n=m}^{\infty} 2^{\alpha n} q_{2^n} \leq 2^{1+\alpha} \sum_{n=m}^{\infty} \sum_{j=2^{n-1}+1}^{2^n} j^{\alpha-1} q_j = 2^{1+\alpha} \sum_{n=2^{m-1}+1}^{\infty} n^{\alpha-1} q_n.$$

On the other hand,

$$2^{\alpha n} q_{2^n} \geq 2^{-\alpha} j^{\alpha} q_j, \quad 2^n \leq j \leq 2^{n+1} - 1, \quad n \geq m,$$

hence

$$2^{\alpha n} q_{2^n} \geq \frac{1}{2^{\alpha} 2^n} \sum_{j=2^n}^{2^{n+1}-1} j^{\alpha} q_j \geq \frac{1}{2^{\alpha}} \sum_{j=2^n}^{2^{n+1}-1} j^{\alpha-1} q_j,$$

and

$$\sum_{n=m}^{\infty} 2^{\alpha n} q_{2^n} \geq \frac{1}{2^{\alpha}} \sum_{n=m}^{\infty} \sum_{j=2^n}^{2^{n+1}-1} j^{\alpha-1} q_j = \frac{1}{2^{\alpha}} \sum_{n=2^m}^{\infty} n^{\alpha-1} q_n.$$

□

8. APPENDIX B: CROFT'S APPROACH

In this section we present a different proof of Corollary 3.8 based on Croft's approach from [8] dealing with quasi-exponential series, that is with trigonometrical series with real frequencies. However, as we remarked above, the argument in [8] seems to be incomplete.

Let us briefly compare Croft's approach with the one of the present paper. For a sufficiently small parameter θ_0 both approaches aim at finding d and γ such that $\delta(\theta_0) = |1 - P_{d,\gamma}(\theta_0)|$ is "small", in particular, $\delta(\theta_0) \leq c\theta_0^{2-\epsilon}$ where a constant c does not depend on θ_0 . Croft proceeds by requiring $\text{Im } P_{d,\gamma}(\theta_0) = 0$. This way he expresses γ in terms of d and then chooses d to ensure the inequality above. Unfortunately, his proof stops at this step. Proceeding in a different way, for a fixed γ we minimize the quadratic function $q(d) = (1+d)^2 |1 - P_{d,\gamma}(\theta_0)|^2$ with respect to d . This relates d to γ , and allows us to make the quantity δ "small" enough to fit the steps of our inductive constructions in Section 3. The two steps lead eventually to similar estimates of $\delta(\theta_0)$. On the other hand, we also have to take care of a) getting polynomials with integer frequencies eventually, b) spreading out our estimates for δ to large sets, c) extending our estimates for polynomials $(Q_m)_{m=1}^{\infty}$ from fixed θ_0 to appropriate sequences of $(\theta_m)_{m=1}^{\infty} \subset [0, \pi]$ going to 0 and then, finally, d) of constructing $f \in \mathcal{A}^+(w)$ out of $(Q_m)_{m=1}^{\infty}$ via a limiting procedure.

Lemma 8.1. *Let*

$$P(\theta) = \sum_{k=1}^n a_k e^{ik\theta} \in \mathcal{A}^+$$

$\epsilon \in (0, 1]$, and $\alpha \in (0, \epsilon/2]$. Then for any $\theta_0 \in (0, 1/n^{1/\alpha}]$,

$$(8.1) \quad 0 < \text{Im } P(\theta_0) \leq \theta_0^{1-\alpha} < 1,$$

and

$$(8.2) \quad 0 \leq 1 - \operatorname{Re} P(\theta_0) \leq \theta_0^{2-\epsilon}.$$

Moreover,

$$(8.3) \quad \|P'\|_\infty \leq \theta_0^{-\alpha}.$$

Proof. Since $n\theta_0 \in (0, 1]$ and $n \leq \theta_0^{-\alpha}$,

$$0 < \frac{2\theta_0}{\pi} \sum_{k=1}^n ka_k \leq \operatorname{Im} P(\theta_0) = \sum_{k=1}^n a_k \sin(k\theta_0) \leq n\theta_0 \leq \theta_0^{1-\alpha} < 1,$$

and, in view of $\epsilon \geq 2\alpha$,

$$0 \leq 1 - \operatorname{Re} P(\theta_0) = \sum_{k=1}^n a_k (1 - \cos(k\theta_0)) \leq n^2 \theta_0^2 / 2 \leq \theta_0^{2(1-\alpha)} \leq \theta_0^{2-\epsilon},$$

so that (8.1) and (8.2) hold. Moreover,

$$|P'(\theta)| \leq \sum_{k=1}^n ka_k \leq n \leq \theta_0^{-\alpha}, \quad |\theta| \leq \pi,$$

and we get (8.3). \square

As an illustration, to show that Croft's idea actually works, we provide now a proof of Corollary 3.8 following Croft's approach.

Proof of Corollary 3.8. For $d > 0$ and $\tau > 0$, define

$$(8.4) \quad Q(\theta) = \frac{P(\theta) + de^{i\tau\theta}}{1+d}.$$

Then

$$1 - Q(\theta) = \frac{(1 - P(\theta)) + d(1 - e^{i\tau\theta})}{1+d}$$

and

$$P(\theta) - Q(\theta) = \frac{d(P(\theta) - e^{i\tau\theta})}{1+d}.$$

Fix $\Theta_0 \in (0, 1)$ such that

$$(8.5) \quad \frac{n}{\varphi^{1/4}(\theta)} \leq 1, \quad \text{and} \quad \nu + \frac{\log \varphi(\theta)}{|\log \theta|} \leq 1, \quad \theta \in (0, \Theta_0],$$

so that, in particular,

$$(8.6) \quad \varphi(\theta) \geq 1 \quad \text{and} \quad \theta^{1-\nu} \varphi(\theta) \leq 1, \quad \theta \in (0, \Theta_0].$$

For $\theta \in (0, \Theta_0]$ define

$$\epsilon(\theta) := \nu + \frac{\log \varphi(\theta)}{|\log \theta|} \in (0, 1], \quad \alpha(\theta) := \frac{\log \varphi(\theta)}{4|\log \theta|} < \frac{\epsilon}{4},$$

and note that

$$\theta^{\epsilon(\theta)} = \frac{\theta^\nu}{\varphi(\theta)}, \quad \theta^{\alpha(\theta)} = \frac{1}{\varphi^{1/4}(\theta)}.$$

Then the assumption $\theta \in (0, 1/n^{1/\alpha(\theta)})$ takes the form $n \leq \varphi^{1/4}(\theta)$ and holds for $\theta \in (0, \Theta_0]$, by (8.5).

Furthermore, for $\theta \in (0, \Theta_0]$ define

$$d(\theta) := \frac{\theta^\nu}{\phi^{1/2}(\theta)} \quad \text{and} \quad v(\theta) := \frac{\operatorname{Im} P(\theta)}{d(\theta)}.$$

By (8.1) and (8.6) we infer that

$$(8.7) \quad 0 < v(\theta) \leq \frac{\theta^{1-\alpha(\theta)}}{d(\theta)} = \theta^{1-\nu} \phi^{3/4}(\theta) \leq \theta^{1-\nu} \phi(\theta) \leq 1,$$

hence

$$(8.8) \quad \tau(\theta) = \frac{2\pi - \arcsin v(\theta)}{\theta}, \quad \theta \in (0, \Theta_0],$$

is well-defined. Observe that τ is continuous on $(0, \Theta_0]$ and it satisfies $\lim_{\theta \rightarrow 0} \tau(\theta) = +\infty$. Therefore, there exists a sequence $(\theta_m)_{m \geq m_0} \subset (0, \Theta_0]$ satisfying

$$(8.9) \quad \tau_m := \tau(\theta_m) = m, \quad m \geq m_0.$$

We may assume that $\theta_{m+1} < \theta_m$, and (8.9) and (8.8) imply that $m\theta_m \leq 2\pi, m \geq m_0$. Moreover, the latter condition implies that $m\theta_m \in (3\pi/2, 2\pi]$ for large m , so without loss of generality, we may assume that $m\theta_m \in (3\pi/2, 2\pi], m \geq m_0$.

Define a polynomial Q_m by (8.4) with $d = d(\theta_m)$ and $\tau = \tau_m = m$. Then, employing (8.8) and (8.9), we obtain that

$$(8.10) \quad (1 + d(\theta_m)) \operatorname{Im} Q_m(\theta_m) = d(\theta_m) [v(\theta_m) + \sin(m\theta_m)] = 0.$$

Moreover, by (8.2),

$$(8.11) \quad 1 - \operatorname{Re} Q_m(\theta_m) \leq \theta_m^{2-\epsilon(\theta_m)} + d(\theta_m)(1 - \cos(m\theta_m)),$$

and, in view of $m\theta_m \in (3\pi/2, 2\pi], m \geq m_0$, and (8.7),

$$1 - \cos(m\theta_m) \leq \sin^2(m\theta_m) = v^2(\theta_m) \leq \theta_m^{2(1-\nu)} \phi^{3/2}(\theta_m).$$

Thus, taking into account (8.11) and (8.10),

$$|1 - Q_m(\theta_m)| \leq \theta_m^{2-\epsilon(\theta_m)} + \frac{\theta_m^\nu}{\phi^{1/2}(\theta_m)} \theta_m^{2(1-\nu)} \phi^{3/2}(\theta_m) = 2\varphi(\theta_m) \theta_m^{2-\nu}.$$

Next, to estimate the distance between P and Q_m we note that

$$\|P - Q_m\|_{\mathcal{A}} \leq d(\theta_m) \frac{\|P - e^{im\theta}\|_{\mathcal{A}}}{1 + d(\theta_m)} \leq 2d(\theta_m) = \frac{2\theta_m^\nu}{\varphi^{1/2}(\theta_m)},$$

and then, by Remark 3.7,

$$\|P - Q_m\|_{\mathcal{A}_\nu} \leq m^\nu \|P - Q_m\|_{\mathcal{A}} \leq \frac{2(2\pi)^\nu}{\varphi^{1/2}(\theta_m)}.$$

Finally, (8.3), (8.5), and (8.6) imply that

$$\begin{aligned}
\|Q'_m\|_\infty &\leq \|P'\|_\infty + d(\theta_m)m \\
&\leq \theta_m^{-\alpha(\theta_m)} + 2\pi \frac{\theta_m^{\nu-1}}{\varphi^{1/2}(\theta_m)} \\
&= \varphi^{1/4}(\theta_m) + 2\pi \frac{\theta_m^{\nu-1}}{\varphi^{1/2}(\theta_m)} \\
&\leq 8 \frac{\theta_m^{\nu-1}}{\varphi^{1/2}(\theta_m)}.
\end{aligned}$$

Therefore, if $|\theta_m - \theta| \leq \varphi^{3/2}(\theta_m)\theta_m^{3-2\nu}$ then

$$\begin{aligned}
|1 - Q_m(\theta)| &\leq 2\varphi(\theta_m)\theta_m^{2-\nu} + 8 \frac{\theta_m^{\nu-1}}{\varphi^{1/2}(\theta_m)} \varphi^{3/2}(\theta_m)\theta_m^{3-2\nu} \\
&= 10\varphi(\theta_m)\theta_m^{2-\nu}.
\end{aligned}$$

□

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